k -Pell, k -Pell–Lucas and modified k -Pell sedenions

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Communicated by H. M. Srivastava

Received August 25, 2017

Accepted November 10, 2017

Published April 24, 2018

The aim of this work is to present the k -Pell, the k -Pell–Lucas and the Modified k -Pell sedenions and we give some properties involving these sequences, including the Binet-style formulae and the ordinary generating functions.

Keywords: k -Pell numbers; k -Pell–Lucas numbers; modified k -Pell numbers; sedenions; generating functions.

AMS Subject Classification: 11B37, 11R52, 05A15

1. Introduction and Background

The set of sedenions, denoted by $\mathbb{S}$, form a 16-dimensional noncommutative and nonassociative algebra over the set of real numbers, obtained by applying the Cayley–Dickson construction to the octonions. Like octonions, multiplication of sedenions is neither commutative nor associative. Sedenions have its applications in many fields of science such as in electromagnetic theory, linear gravity and in the field of quantum mechanics (see [17]).

A sedenion s is defined by

$$s = \sum_{i=0}^{15} a_i e_i,$$

where $a_0, a_1, \dots, a_{15} \in \mathbb{R}$ and $e_0, e_1, \dots, e_{15}$ are what we call *unit sedenions* such that e_0 is the unit element and $e_1, \dots, e_{15}$ are imaginaries satisfying, for $i, j, k = 1, \dots, 15$ the following multiplication rules:

$$e_0 e_i = e_i e_0 = e_i, \quad (e_i)^2 = -e_0, \quad (1.1)$$

$$e_i e_j = -e_j e_i, \quad i \neq j, \quad (1.2)$$

$$e_i (e_j e_k) = -(e_i e_j) e_k, \quad i \neq j, \quad e_i e_j \neq \pm e_k. \quad (1.3)$$

Throughout the text, we shall consider that $\{e_i \in \mathbb{S} : i = 0, 1, \dots, 15\}$ is the canonical basis of $\mathbb{S}$.

As found in [1] and [15], a sedenion can also be considered as an ordered pair of octonions

$$s = (o_1, o_2), \quad (1.4)$$

where o_1, o_2 are elements of the octonion algebra $\mathbb{O}$. Note that $\mathbb{S}$ is the double of $\mathbb{O}$ and the indices $i = 0, 1, \dots, 7$ correspond to the octonion base elements, while those where $i = 8, \dots, 15$ correspond to the pure sedenion base elements (see [10]).

Addition of sedenions is done component-wise and multiplication of sedenions is defined by bilinearity and the multiplication rule of the base elements (see [10]). Thus, if $s = \sum_{i=0}^{15} a_i e_i$, $t = \sum_{j=0}^{15} b_j e_j \in \mathbb{S}$, we have

$$st = \sum_{i,j=0}^{15} a_i b_j (e_i e_j), \quad (1.5)$$

where $e_i e_j$ satisfies the identities (1.1), (1.2) and (1.3). Curiously, in [3], the authors introduce an efficient algorithm for the multiplication of sedenions by showing how to compute a sedenions product with 120 real multiplications and 344 real additions.

In the literature several research has been dedicated to the sequences of positive integers (also defined recursively). One of such type of sequences that have been studied over several years is the well-known Fibonacci (and Lucas) sequence. Many papers are dedicated to the Fibonacci sequence, such as the works of Hoggatt, in [13], Vorobiov, in [18], and Koshy, in [16], among others. Also a great investigation is dedicated to their generalizations (see, for example, [2, 4, 6, 12], among other works). For background about the Fibonacci and Lucas numbers, the reader can find their properties in the books [13] and [18]. The Fibonacci numbers F_n are the terms of the sequence $\{0, 1, 1, 2, 3, 5, \dots\}$ where each term is the sum of the two preceding terms, that is,

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

beginning with the values $F_0 = 0$ and $F_1 = 1$. We can extend the sequence $\{F_n\}_{n \in \mathbb{N}}$ to a negative n (see, for example, [19]), such that $F_{-n} = (-1)^{n-1} F_n$. One example of a generalization of the Fibonacci sequence is the sequence of k -Fibonacci numbers $F_{k,n}$ introduced in [12] and defined by

$$F_{k,n} = k F_{k,n-1} + F_{k,n-2}, \quad n \geq 2, \quad k \in \mathbb{N},$$

with the initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$. In particular, for $k = 1$ and $k = 2$ we obtain the classic Fibonacci numbers and the Pell numbers, respectively. Recall

that, the Pell sequence is defined by the following recurrence relation:

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1}, \quad n \geq 1.$$

This sequence has been studied and some of its basic properties are known (see, for example, the study of Horadam, in [14]). One generalization of the Pell sequence is the *k*-Pell sequence for any positive integer number *k*.

The *k*-Pell sequence $\{P_{k,n}\}_n$ is defined recursively (in a similar way as in the previous sequence) by

$$P_{k,0} = 0, \quad P_{k,1} = 1, \quad P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \quad n \geq 1. \quad (1.6)$$

For more details about this sequence, see [7] and [9]. The Pell–Lucas sequence $\{Q_n\}_n$ is defined by

$$Q_0 = Q_1 = 2, \quad Q_{n+1} = 2Q_n + Q_{n-1}, \quad n \geq 1, \quad (1.7)$$

and the *k*-Pell–Lucas sequence $\{Q_{k,n}\}_n$ is given recursively by

$$Q_{k,0} = Q_{k,1} = 2, \quad Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1}, \quad n \geq 1. \quad (1.8)$$

For more details about this sequence see, for example, [11].

Another sequence of integers related with these sequences is the Modified Pell sequence $\{q_n\}_n$ defined by

$$q_0 = q_1 = 1, \quad q_{n+1} = 2q_n + q_{n-1}, \quad n \geq 1. \quad (1.9)$$

A generalization of the previous sequence is the sequence $\{q_{k,n}\}_n$ of Modified *k*-Pell numbers satisfying the recursive sequence given by

$$q_{k,0} = q_{k,1} = 1, \quad q_{k,n+1} = 2q_{k,n} + kq_{k,n-1}, \quad n \geq 1. \quad (1.10)$$

The Binet-style formulae for these sequences are given by

$$P_{k,n} = \frac{(r_1)^n - (r_2)^n}{r_1 - r_2}, \quad Q_{k,n} = (r_1)^n + (r_2)^n, \quad q_{k,n} = \frac{(r_1)^n + (r_2)^n}{2}, \quad (1.11)$$

respectively, where $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation $r^2 - 2r - k = 0$ associated with the above recurrence relations (1.6), (1.8) and (1.10). Note that $r_1 + r_2 = 2$, $r_1 r_2 = -k$ and $r_1 - r_2 = 2\sqrt{1+k}$.

For more details about these sequences see, for example, [5, 8, 9] and [11].

The more recent research in the topic of sequences of sedenions is the work of Bilgili *et al.* in [1] about the Fibonacci and Lucas sedenions. Motivated essentially by this work, in this paper, we introduce the *k*-Pell, *k*-Pell–Lucas and the Modified *k*-Pell sedenions and we give some properties of them. The next three sections are dedicated to present some properties involving these sequences, including the Binet-style formulae and the ordinary generating functions for each one of these sequences of sedenions.

2. The *k*-Pell, *k*-Pell–Lucas and Modified *k*-Pell Sedenions

This section aims to set out the definition of the *k*-Pell, *k*-Pell–Lucas and Modified *k*-Pell sedenions and some elementary results. Following [1], the *n*th *k*-Pell sedenion

$\{\text{SP}_{k,n}\}$, k -Pell–Lucas sedenion $\{\text{SPL}_{k,n}\}$ and Modified k -Pell sedenion $\{\text{SMP}_{k,n}\}$ are defined by

$$\text{SP}_{k,n} = \sum_{i=0}^{15} P_{k,n+i} e_i, \quad (2.1)$$

$$\text{SPL}_{k,n} = \sum_{i=0}^{15} Q_{k,n+i} e_i, \quad (2.2)$$

$$\text{SMP}_{k,n} = \sum_{i=0}^{15} q_{k,n+i} e_i, \quad (2.3)$$

respectively.

Note that $Q_{k,n} = 2q_{k,n}$ and then we can connect the sedenions (2.2) and (2.3) by

$$\text{SPL}_{k,n} = 2\text{SMP}_{k,n}.$$

It is easy to show that $P_{k,-n} = -(-k)^{-n} P_{k,n}$, $Q_{k,-n} = (-k)^{-n} Q_{k,n}$ and $q_{k,-n} = \frac{1}{2} Q_{k,-n}$, then we immediately have that the k -Pell, k -Pell–Lucas and Modified k -Pell sedenions with negative indexes are, respectively

$$\text{SP}_{k,-n} = \sum_{i=0}^{15} -(-k)^{i-n} P_{k,n-i} e_i, \quad \text{SPL}_{k,-n} = \sum_{i=0}^{15} (-k)^{i-n} Q_{k,n-i} e_i,$$

and

$$\text{SMP}_{k,-n} = \sum_{i=0}^{15} (-k)^{i-n} q_{k,n-i} e_i.$$

Note that the sequences $\{\text{SP}_{k,n}\}$, $\{\text{SPL}_{k,n}\}$ and $\{\text{SMP}_{k,n}\}$ of k -Pell, k -Pell–Lucas and Modified k -Pell sedenions satisfy, respectively, the following second-order recursive relations:

$$\text{SP}_{k,n+1} = 2\text{SP}_{k,n} + k\text{SP}_{k,n-1} \quad (2.4)$$

with initial conditions $\text{SP}_{k,0} = \sum_{i=0}^{15} P_{k,i} e_i$ and $\text{SP}_{k,1} = \sum_{i=0}^{15} P_{k,i+1} e_i$,

$$\text{SPL}_{k,n+1} = 2\text{SPL}_{k,n} + k\text{SPL}_{k,n-1} \quad (2.5)$$

with initial conditions $\text{SPL}_{k,0} = \sum_{i=0}^{15} Q_{k,i} e_i$ and $\text{SPL}_{k,1} = \sum_{i=0}^{15} Q_{k,i+1} e_i$, and

$$\text{SMP}_{k,n+1} = 2\text{SMP}_{k,n} + k\text{SMP}_{k,n-1} \quad (2.6)$$

with initial conditions $\text{SMP}_{k,0} = \sum_{i=0}^{15} q_{k,i} e_i$ and $\text{SMP}_{k,1} = \sum_{i=0}^{15} q_{k,i+1} e_i$.

Now, using the identity $Q_{k,n} = 2q_{k,n}$, the [5, Propositions 3 and 4] and the identities (2.1), (2.2) and (2.3), we easily obtain the following results:

Proposition 2.1. *For a natural number n and a positive integer k , if $\text{SP}_{k,n}$, $\text{SPL}_{k,n}$ and $\text{SMP}_{k,n}$ are, respectively, the k -Pell, k -Pell–Lucas and Modified k -Pell*

sedenions, then the following identities are true:

- (1) $\text{SPL}_{k,n} = 2\text{SMP}_{k,n}$
- (2) $\text{SP}_{k,n} = \frac{1}{2(k+1)}(\text{SPL}_{k,n+1} - \text{SPL}_{k,n})$
- (3) $\text{SP}_{k,n} = \frac{1}{k+1}(\text{SMP}_{k,n+1} - \text{SMP}_{k,n})$
- (4) $\text{SPL}_{k,n} = 2(\text{SP}_{k,n+1} - \text{SP}_{k,n})$
- (5) $\text{SPL}_{k,n+1} = 2(\text{SP}_{k,n+1} + k\text{SP}_{k,n})$.

3. Generating Functions and Binet Formulas of these Sequences

Next, we shall give the ordinary generating functions for the *k*-Pell, the *k*-Pell–Lucas and the Modified *k*-Pell sedenion sequences. We shall write each sequence as a power series where each term of the sequences correspond to coefficients of the series. Consider the *k*-Pell sedenion sequences $\{\text{SP}_{k,n}\}$, the *k*-Pell–Lucas sedenion sequences $\{\text{SPL}_{k,n}\}$ and the Modified *k*-Pell sedenion sequences $\{\text{SMP}_{k,n}\}$. By definition of ordinary generating function of a sequence, considering these sequences, the associated ordinary generating function $g_{\text{SP}_{k,n}}(t)$, $g_{\text{SPL}_{k,n}}(t)$ and $g_{\text{SMP}_{k,n}}(t)$ are defined, respectively, by

$$g_{\text{SP}_{k,n}}(t) = \sum_{n=0}^{\infty} \text{SP}_{k,n} t^n, \quad (3.1)$$

$$g_{\text{SPL}_{k,n}}(t) = \sum_{n=0}^{\infty} \text{SPL}_{k,n} t^n \quad (3.2)$$

and

$$g_{\text{SMP}_{k,n}}(t) = \sum_{n=0}^{\infty} \text{SMP}_{k,n} t^n. \quad (3.3)$$

We obtain the following result:

Theorem 3.1. *The ordinary generating function for the *k*-Pell, the *k*-Pell–Lucas and the Modified *k*-Pell sedenion sequences are, respectively*

- (1) $g_{\text{SP}_{k,n}}(t) = \frac{\text{SP}_{k,0} + (\text{SP}_{k,1} - 2\text{SP}_{k,0})t}{1 - 2t - kt^2},$
- (2) $g_{\text{SPL}_{k,n}}(t) = \frac{\text{SPL}_{k,0} + (\text{SPL}_{k,1} - 2\text{SPL}_{k,0})t}{1 - 2t - kt^2},$
- (3) $g_{\text{SMP}_{k,n}}(t) = \frac{\text{SMP}_{k,0} + (\text{SMP}_{k,1} - 2\text{SMP}_{k,0})t}{1 - 2t - kt^2}.$

Proof. (1) By (3.1), we have

$$g_{\text{SP}_{k,n}}(t) = \text{SP}_{k,0} + \text{SP}_{k,1}t + \text{SP}_{k,2}t^2 + \cdots + \text{SP}_{k,n}t^n + \cdots \quad (3.4)$$

Multiplying both sides of (3.4) by $-2t$, we obtain

$$-2tg_{\text{SP}_{k,n}}(t) = -2\text{SP}_{k,0}t - 2\text{SP}_{k,1}t^2 - 2\text{SP}_{k,2}t^3 - \cdots - 2\text{SP}_{k,n}t^{n+1} - \cdots \quad (3.5)$$

Now multiplying both sides of (3.4) by $-kt^2$, we get

$$\begin{aligned} -kt^2 g_{\text{SP}_{k,n}}(t) &= -k\text{SP}_{k,0}t^2 - k\text{SP}_{k,1}t^3 \\ &\quad - k\text{SP}_{k,2}t^4 - \dots - k\text{SP}_{k,n}t^{n+2} - \dots \end{aligned} \quad (3.6)$$

Adding (3.4), (3.5), (3.6) and using (2.4), we have

$$(1 - 2t - kt^2)g_{\text{SP}_{k,n}}(t) = \text{SP}_{k,0} + (\text{SP}_{k,1} - 2\text{SP}_{k,0})t$$

and the result follows.

In a similar way, we can obtain the ordinary generating functions stated in (2) and (3). $\square$

The next result gives the Binet-style formulae for the k -Pell, the k -Pell–Lucas and the Modified k -Pell sedenion sequences. Note that these Binet-style formulae are similar to those considered in (1.11).

Theorem 3.2. (*The Binet-style formulae*) For $n \geq 0$, we have

- (1) $\text{SP}_{k,n} = \frac{\widehat{r}_1(r_1)^n - \widehat{r}_2(r_2)^n}{r_1 - r_2},$
- (2) $\text{SPL}_{k,n} = \widehat{r}_1(r_1)^n + \widehat{r}_2(r_2)^n,$
- (3) $\text{SMP}_{k,n} = \frac{\widehat{r}_1(r_1)^n + \widehat{r}_2(r_2)^n}{2},$

where $\widehat{r}_1$ and $\widehat{r}_2$ are sedenions defined by $\widehat{r}_1 = \sum_{i=0}^{15} (r_1)^i e_i$ and $\widehat{r}_2 = \sum_{i=0}^{15} (r_2)^i e_i$, respectively.

Proof. (1) Using (2.1) and the Binet formula for the k -Pell numbers considered in (1.11), we have

$$\begin{aligned} \text{SP}_{k,n} &= \sum_{i=0}^{15} \left(\frac{(r_1)^{n+i} - (r_2)^{n+i}}{r_1 - r_2} \right) e_i \\ &= \frac{1}{r_1 - r_2} \left((r_1)^n \sum_{i=0}^{15} (r_1)^i e_i - (r_2)^n \sum_{i=0}^{15} (r_2)^i e_i \right) \end{aligned}$$

and the result easily follows.

The proof of (2) and (3) is similar by the use of the correspondent identities and results involving the k -Pell–Lucas and Modified k -Pell numbers. $\square$

4. Some Identities Involving these Sequences

According with recurrence relations (1.6), (1.8) and (1.10) and using the well-known results involving recursive sequences, consider the respective characteristic equation and note that $r_1 r_2 = -k$, $r_1 - r_2 = 2\sqrt{1+k}$. As a consequence of the Binet formulae

of Theorem 3.2, we get for these sequences of sedenions the following interesting identities.

Proposition 4.1. (*Catalan's identities*) *For natural numbers n, r , with $n \geq r$ and a positive integer k , if $\text{SP}_{k,n}$, $\text{SPL}_{k,n}$ and $\text{SMP}_{k,n}$ are, respectively, the n th k -Pell, k -Pell–Lucas and Modified k -Pell sedenions, then the following identities are true:*

$$\begin{aligned}\text{SP}_{k,n-r}\text{SP}_{k,n+r} - (\text{SP}_{k,n})^2 &= (-k)^{n-r} \left(\left(\frac{-s_2(r_2)^{2r} - s_1(r_1)^{2r}}{4(k+1)} \right) - (\text{SP}_{k,r})^2 \right), \\ \text{SPL}_{k,n-r}\text{SPL}_{k,n+r} - (\text{SPL}_{k,n})^2 &= (-k)^{n-r} (t_1(r_1)^{2r} + t_2(r_2)^{2r} - (\text{SPL}_{k,r})^2), \\ \text{SMP}_{k,n-r}\text{SMP}_{k,n+r} - (\text{SMP}_{k,n})^2 &= \frac{(-k)^{n-r}}{4} (t_1(r_1)^{2r} + t_2(r_2)^{2r} - (2\text{SMP}_{k,r})^2),\end{aligned}$$

where $\hat{r}_1$ and $\hat{r}_2$ are the sedenions defined in Theorem 3.2 and s_1, s_2, t_1, t_2 are sedenions defined by $s_1 = \hat{r}_2\hat{r}_1 - (\hat{r}_1)^2$, $s_2 = \hat{r}_1\hat{r}_2 - (\hat{r}_2)^2$, $t_1 = s_1 + 2(\hat{r}_1)^2$ and $t_2 = s_2 + 2(\hat{r}_2)^2$, respectively.

Proof. For the first identity, using the first Binet formula of Theorem 3.2 and the fact that $r_1r_2 = -k$, $r_1 - r_2 = 2\sqrt{1+k}$, $\frac{r_1}{r_2} = \frac{(r_1)^2}{-k}$, $\frac{r_2}{r_1} = \frac{(r_2)^2}{-k}$,

$$\begin{aligned}\text{SP}_{k,n-r}\text{SP}_{k,n+r} - (\text{SP}_{k,n})^2 &= \left(\frac{\hat{r}_1(r_1)^{n-r} - \hat{r}_2(r_2)^{n-r}}{r_1 - r_2} \right) \left(\frac{\hat{r}_1(r_1)^{n+r} - \hat{r}_2(r_2)^{n+r}}{r_1 - r_2} \right) - \left(\frac{\hat{r}_1(r_1)^n - \hat{r}_2(r_2)^n}{r_1 - r_2} \right)^2 \\ &= \frac{(-k)^n \left(-\hat{r}_1\hat{r}_2\left(\frac{r_2}{r_1}\right)^r - \hat{r}_2\hat{r}_1\left(\frac{r_1}{r_2}\right)^r + \hat{r}_1\hat{r}_2 + \hat{r}_2\hat{r}_1 \right)}{(r_1 - r_2)^2} \\ &= \frac{(-k)^{n-r} (-\hat{r}_1\hat{r}_2(r_2)^{2r} - \hat{r}_2\hat{r}_1(r_1)^{2r} + \hat{r}_1\hat{r}_2(r_1r_2)^r + \hat{r}_2\hat{r}_1(r_1r_2)^r)}{(r_1 - r_2)^2} \\ &= (-k)^{n-r} \left(-(\text{SP}_{k,r})^2 + \frac{-(r_2)^{2r}(\hat{r}_1\hat{r}_2 - (\hat{r}_2)^2) - (r_1)^{2r}(\hat{r}_2\hat{r}_1 - (\hat{r}_1)^2)}{(r_1 - r_2)^2} \right) \\ &= (-k)^{n-r} \left(\left(\frac{-s_2(r_2)^{2r} - s_1(r_1)^{2r}}{4(k+1)} \right) - (\text{SP}_{k,r})^2 \right)\end{aligned}$$

and then the result follows.

Using a similar method, by using the respective Binet formula, we can prove the Catalan identity for the k -Pell–Lucas sedenions. Note that using the first identity of Proposition 2.1, we easily find the Catalan identity for the Modified k -Pell sedenions sequences. $\square$

Observe that for $r = 1$ in Catalan's identities obtained, we get the Cassini identities for these sequences.

Proposition 4.2. (*Cassini's identities*) *For a natural number n and a positive integer k , if $\text{SP}_{k,n}$, $\text{SPL}_{k,n}$ and $\text{SMP}_{k,n}$ are, respectively, the n th k -Pell, k -Pell–Lucas*

and Modified k -Pell sedenions, then the following identities are true:

$$\begin{aligned} \text{SP}_{k,n-1}\text{SP}_{k,n+1} - (\text{SP}_{k,n})^2 &= (-k)^{n-1} \left(\left(\frac{-s_2(r_2)^2 - s_1(r_1)^2}{4(k+1)} \right) - (\text{SP}_{k,1})^2 \right), \\ \text{SPL}_{k,n-1}\text{SPL}_{k,n+1} - (\text{SPL}_{k,n})^2 &= (-k)^{n-1} (t_1(r_1)^2 + t_2(r_2)^2 - (\text{SPL}_{k,1})^2), \\ \text{SMP}_{k,n-1}\text{SMP}_{k,n+1} - (\text{SMP}_{k,n})^2 &= \frac{(-k)^{n-1}}{4} (t_1(r_1)^2 + t_2(r_2)^2 - (2\text{SMP}_{k,1})^2), \end{aligned}$$

where $\hat{r}_1$ and $\hat{r}_2$ are the sedenions defined in Theorem 3.2 and s_1, s_2, t_1, t_2 are sedenions defined by $s_1 = \hat{r}_2\hat{r}_1 - (\hat{r}_1)^2$, $s_2 = \hat{r}_1\hat{r}_2 - (\hat{r}_2)^2$, $t_1 = s_1 + 2(\hat{r}_1)^2$ and $t_2 = s_2 + 2(\hat{r}_2)^2$, respectively.

The d'Ocagne identity for each of these sequences can also be obtained using the Binet formula for these type of sequences.

Proposition 4.3. (*d'Ocagne's identities*) Suppose that n is a nonnegative integer number and m any natural number. If $m > n$ and $\text{SP}_{k,n}$, $\text{SPL}_{k,n}$ and $\text{SMP}_{k,n}$ are, respectively, the n th k -Pell, k -Pell-Lucas and Modified k -Pell sedenions, then the expressions of the d'Ocagne's identities are given by

- (1) $(-k)^n (\text{SP}_{k,m-n} + \frac{1}{2\sqrt{k+1}}(d_1(r_1)^{m-n} - d_2(r_2)^{m-n}))$,
- (2) $(-k)^n 2\sqrt{1+k}(\text{SPL}_{k,m-n} + d_2(r_2)^{m-n} - (d_1 + 2\hat{r}_1)(r_1)^{m-n})$,
- (3) $\frac{(-k)^n \sqrt{1+k}}{2}(2\text{SMP}_{k,m-n} + d_2(r_2)^{m-n} - (d_1 + 2\hat{r}_1)(r_1)^{m-n})$,

respectively, where $\hat{r}_1$ and $\hat{r}_2$ are the sedenions defined in Theorem 3.2 and d_1, d_2 are sedenions defined by $d_1 = \hat{r}_1\hat{r}_2 - \hat{r}_1$, $d_2 = \hat{r}_2\hat{r}_1 - \hat{r}_2$, respectively.

Proof. Once more, using the first Binet formula of Theorem 3.2 and the fact that $r_1r_2 = -k$, $r_1 - r_2 = 2\sqrt{1+k}$,

$$\begin{aligned} &\text{SP}_{k,m}\text{SP}_{k,n+1} - \text{SP}_{k,m+1}\text{SP}_{k,n} \\ &= \left(\frac{\hat{r}_1(r_1)^m - \hat{r}_2(r_2)^m}{r_1 - r_2} \right) \left(\frac{\hat{r}_1(r_1)^{n+1} - \hat{r}_2(r_2)^{n+1}}{r_1 - r_2} \right) \\ &\quad - \left(\frac{\hat{r}_1(r_1)^{m+1} - \hat{r}_2(r_2)^{m+1}}{r_1 - r_2} \right) \left(\frac{\hat{r}_1(r_1)^n - \hat{r}_2(r_2)^n}{r_1 - r_2} \right) \\ &= \frac{(-k)^n (\hat{r}_1\hat{r}_2(r_1)^{m-n}(r_1 - r_2) - \hat{r}_2\hat{r}_1(r_2)^{m-n}(r_1 - r_2))}{(r_1 - r_2)^2} \\ &= \frac{(-k)^n (r_1 - r_2)(\hat{r}_1\hat{r}_2(r_1)^{m-n} - \hat{r}_2\hat{r}_1(r_2)^{m-n})}{(r_1 - r_2)^2} \\ &= (-k)^n \left(\text{SP}_{k,m-n} + \frac{(r_1)^{m-n}(\hat{r}_1\hat{r}_2 - \hat{r}_1) + (r_2)^{m-n}(\hat{r}_2 - \hat{r}_2\hat{r}_1)}{(r_1 - r_2)} \right) \\ &= (-k)^n \left(\text{SP}_{k,m-n} + \frac{1}{2\sqrt{k+1}}(d_1(r_1)^{m-n} - d_2(r_2)^{m-n}) \right) \end{aligned}$$

and then the result follows.

Analogously, by using the respective Binet formula we can prove the d’Ocagne’s identity for the k -Pell–Lucas sedenions. Note that using the first identity of Proposition 2.1, we easily find the d’Ocagne’s identity for the Modified k -Pell sedenions sequences. $\square$

5. Conclusions

In this paper, the sequence of k -Pell sedenions defined by a recurrence relation of second order was introduced. In the same way, we introduce the sequence of k -Pell–Lucas sedenions and the sequence of Modified k -Pell sedenions. Some properties involving these sequences, including the Binet-style formulae and the ordinary generating functions, were presented. Note that the results stated in this paper for the particular case of $k = 1$ give us the correspondent results for the sequences of the Pell, the Pell–Lucas and the Modified Pell sedenions.

Acknowledgment

The research of the author was financed by Portuguese Funds through FCT — Fundação para a Ciência e a Tecnologia, within the Projects UID/MAT/00013/2013 and UID/CED/00194/2013. Also, the author would like to thank the referees for their pertinent comments and valuable suggestions, which significantly improved the manuscript.

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