

Article

On the Bicomplex Generalized Tribonacci Quaternions

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Abstract: In this paper, we introduce the bicomplex generalized tribonacci quaternions. Furthermore, Binet's formula, generating functions, and the summation formula for this type of quaternion are given. Lastly, as an application, we present the determinant of a special matrix, and we show that the determinant is equal to the n^{th} term of the bicomplex generalized tribonacci quaternions.

Keywords: bicomplex number; generalized tribonacci sequence; bicomplex generalized tribonacci quaternion

MSC: 11B39; 20G20; 11R52

1. Introduction

Corrado Segre introduced bicomplex numbers in 1892. The bicomplex numbers are defined by four elements $1, i, j, ij$, where i, j and ij satisfy the properties:

$$i^2 = -1, j^2 = -1, ij = ji. \quad (1)$$

Thus, a bicomplex number q can be expressed as follows:

$$q = q_1 + iq_2 + jq_3 + ijq_4 = q_1 + iq_2 + j(q_3 + iq_4),$$

where $q_1, q_2, q_3, q_4 \in \mathbb{R}$.

For any $q = q_1 + iq_2 + jq_3 + ijq_4$ and $p = p_1 + ip_2 + jp_3 + ijp_4$, the addition, subtraction, and multiplication of the bicomplex quaternions are defined by the following:

$$q + p = q_1 + p_1 + i(q_2 + p_2) + j(q_3 + p_3) + ij(q_4 + p_4),$$

$$q - p = q_1 - p_1 + i(q_2 - p_2) + j(q_3 - p_3) + ij(q_4 - p_4),$$

$$q \times p = q_1p_1 - q_2p_2 - q_3p_3 + q_4p_4 + i(q_1p_2 + q_2p_1 - q_3p_4 - q_4p_3) \\ + j(q_1p_3 + q_3p_1 - q_2p_4 - q_4p_2) + ij(q_1p_4 + q_4p_1 + q_2p_3 + q_3p_2),$$

respectively.

Bicomplex numbers have three different conjugations as follows:

$$q_i^* = q_1 - iq_2 + jq_3 - ijq_4,$$

$$q_j^* = q_1 + iq_2 - jq_3 - ijq_4,$$

$$q_{ij}^* = q_1 - iq_2 - jq_3 + ijq_4.$$

For more details about the bicomplex numbers, the readers can refer to [1–4].

On the other hand, the Irish mathematician William Rowan Hamilton introduced quaternions as an extension of complex numbers. A quaternion q is defined in the form:

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where q_0, q_1, q_2, q_3 are real numbers and $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard orthonormal basis in \mathbb{R}^4 , which satisfy the quaternion multiplication rules as:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,$$

$$\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}.$$

Immediately, we note that quaternions and bicomplex numbers are generalizations of complex numbers, but one of the important differences between them is that quaternions are non-commutative, whereas bicomplex numbers are commutative. Furthermore, the bicomplex quaternions can be defined by the basis $\mathbf{1}, \mathbf{i}, \mathbf{j}$ and \mathbf{ij} that satisfy the rules (1).

The n^{th} Fibonacci and Lucas quaternions were defined by Horadam for $n \geq 0$,

$$Q_n = F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k},$$

$$K_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k},$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively [5]. There are many generalizations of the Fibonacci, and Fibonacci-like quaternions have been studied in the literature (see for example [5–18]).

The generalized tribonacci sequence, $\{T_n = T_n(r, s, t; a, b, c)\}_{n \geq 0}$, is defined by the following recurrence relation:

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3} \tag{2}$$

where $T_0 = a, T_1 = b, T_2 = c$ are arbitrary integers and r, s, t are real numbers. The Binet formula for the generalized tribonacci sequence is given by the following relationship:

$$T_n = \frac{A\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\beta^n}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\gamma^n}{(\alpha - \gamma)(\beta - \gamma)}, \tag{3}$$

where $A = c - (\beta + \gamma)b + \beta\gamma a, B = c - (\alpha + \gamma)b + \alpha\gamma a$ and $C = c - (\alpha + \beta)b + \alpha\beta a$. The following are some special cases of the generalized tribonacci sequence $T_n(r, s, t; a, b, c)$:

1. for $r = s = t = 1$ and $a = 0, b = c = 1$, the generalized tribonacci sequence $T_n(1, 1, 1; 0, 1, 1)$ becomes the tribonacci sequence T_n ;
2. for $r = s = t = 1$ and $a = 3, b = 1, c = 3$, the generalized tribonacci sequence $T_n(1, 1, 1; 3, 1, 3)$ becomes the tribonacci–Lucas sequence K_n ;
3. for $r = 0, s = t = 1$ and $a = 3, b = 0, c = 2$, the generalized tribonacci sequence $T_n(0, 1, 1; 3, 0, 2)$ becomes the Perrin sequence Q_n ;
4. for $r = 0, s = t = 1$ and $a = b = c = 1$, the generalized tribonacci sequence $T_n(0, 1, 1; 1, 1, 1)$ becomes the Padovan (Cordonnier) sequence P_n ;
5. for $r = 0, s = t = 1$ and $a = 1, b = 0, c = 1$, the generalized tribonacci sequence $T_n(0, 1, 1; 1, 0, 1)$ becomes the Van der Laan sequence R_n ;
6. for $r = 1, s = 0, t = 1$ and $a = 0, b = c = 1$, the generalized tribonacci sequence $T_n(1, 0, 1; 0, 1, 1)$ becomes the Narayana sequence B_n ;

7. for $r = s = 1, t = 2$ and $a = 0, b = c = 1$, the generalized tribonacci sequence $T_n(1, 1, 2; 0, 1, 1)$ becomes the third-order Jacobsthal sequence $J_n^{(3)}$;
8. for $r = s = 1, t = 2$ and $a = 2, b = 1, c = 5$, the generalized tribonacci sequence $T_n(1, 1, 2; 2, 1, 5)$ becomes the third-order Jacobsthal–Lucas sequence $j_n^{(3)}$;

Therefore, the generalized tribonacci sequences are the generalization of the well-known sequences, which have been studied extensively in the literature (see [19–27]).

The tribonacci quaternion sequences and some generalizations have been studied by many researchers in their papers (see for example [9]).

In [28], the authors defined the bicomplex Fibonacci and Lucas numbers. Moreover, they gave some algebraic properties of these numbers. In [29], Aydın studied the bicomplex Fibonacci quaternions. Recently, in [30], Catarino defined the bicomplex k -Pell quaternions and examined some properties involving these quaternions.

Motivated by the above papers, in this paper, we introduce the bicomplex generalized tribonacci quaternion, and we obtain Binet’s formula, generating functions, and summation formula, as well as some other properties. We explain that the new bicomplex generalized tribonacci quaternions include the bicomplex tribonacci quaternions, the bicomplex tribonacci–Lucas quaternions, the bicomplex Perrin quaternions, the bicomplex Padovan quaternions, the bicomplex Van der Laan quaternions, the bicomplex Narayana quaternions, the bicomplex third-order Jacobsthal quaternions, and the bicomplex third-order Jacobsthal–Lucas quaternions, respectively.

2. Bicomplex Generalized Tribonacci Quaternions

In this section, we introduce the bicomplex generalized tribonacci quaternions.

Definition 1. The bicomplex generalized tribonacci quaternions $\{\mathbb{BCT}_n\}_{n=0}^\infty$ are defined by the following recurrence relation:

$$\mathbb{BCT}_n = T_n + iT_{n+1} + jT_{n+2} + ijT_{n+3}, \tag{4}$$

where T_n is the generalized tribonacci quaternion and $1, i, j, ij$ satisfy the rules (1).

From (2), taking into account (4), we have the recurrence relation of the bicomplex generalized tribonacci quaternions as follows:

$$\mathbb{BCT}_{n+2} = r\mathbb{BCT}_{n+1} + s\mathbb{BCT}_n + t\mathbb{BCT}_{n-1}, \quad n \geq 1 \tag{5}$$

with the initial conditions $\mathbb{BCT}_0 = a + ib + jc + ij(rc + sb + ta)$, $\mathbb{BCT}_1 = b + ic + j(rc + sb + ta) + ij((r^2 + s)c + (t + rs)b + rta)$, and $\mathbb{BCT}_2 = c + i(rc + sb + ta) + j((r^2 + s)c + (t + rs)b + rta) + ij((r^3 + 2rst + t)c + (r^2s + s^2 + rt)b + (r^2t + st)a)$.

For two bicomplex generalized tribonacci quaternions \mathbb{BCT}_n and \mathbb{BCT}_m , addition, subtraction, and multiplication are defined by the following:

$$\mathbb{BCT}_n \pm \mathbb{BCT}_m = T_n \pm T_m + i(T_{n+1} \pm T_{m+1}) + j(T_{n+2} \pm T_{m+2}) + ij(T_{n+3} \pm T_{m+3}),$$

$$\begin{aligned} \mathbb{BCT}_n \times \mathbb{BCT}_m &= (T_n T_m - T_{n+1} T_{m+1} - T_{n+2} T_{m+2} + T_{n+3} T_{m+3}) \\ &\quad + i(T_n T_{m+1} + T_{n+1} T_m - T_{n+2} T_{m+3} - T_{n+3} T_{m+2}) \\ &\quad + j(T_n T_{m+2} + T_{n+2} T_m - T_{n+1} T_{m+3} - T_{n+3} T_{m+1}) \\ &\quad + ij(T_n T_{m+3} + T_{n+3} T_m + T_{n+1} T_{m+2} + T_{n+2} T_{m+1}) \\ &= \mathbb{BCT}_m \times \mathbb{BCT}_n. \end{aligned}$$

The multiplication of a bicomplex generalized tribonacci quaternions by the real scalar λ is defined as follows:

$$\lambda \mathbb{BCT}_n = \lambda T_n + i\lambda T_{n+1} + j\lambda T_{n+2} + ij\lambda T_{n+3}.$$

Furthermore, three different conjugations for the bicomplex generalized tribonacci quaternions are presented by the following:

$$(\mathbb{BCT}_n)_i^* = T_n - iT_{n+1} + jT_{n+2} - ijT_{n+3},$$

$$(\mathbb{BCT}_n)_j^* = T_n + iT_{n+1} - jT_{n+2} - ijT_{n+3},$$

$$(\mathbb{BCT}_n)_{ij}^* = T_n - iT_{n+1} - jT_{n+2} + ijT_{n+3}.$$

Now, we define the generating function:

$$GF_{\mathbb{BCT}_n}(x) = \sum_{n=0}^{\infty} \mathbb{BCT}_n x^n.$$

Theorem 1. The generating function for the bicomplex generalized tribonacci quaternions sequence is as follows:

$$\frac{\mathbb{BCT}_0 + (\mathbb{BCT}_1 - r\mathbb{BCT}_0)x + (\mathbb{BCT}_2 - r\mathbb{BCT}_1 - s\mathbb{BCT}_0)x^2}{1 - rx - sx^2 - tx^3}.$$

Proof. Using the definition of generating function:

$$GF_{\mathbb{BCT}_n}(x) = \mathbb{BCT}_0 + \mathbb{BCT}_1 x + \dots + \mathbb{BCT}_n x^n + \dots \tag{6}$$

Multiplying both sides of (6) by $(1 - rx - sx^2 - tx^3)$ and using (5), we have:

$$(1 - rx - sx^2 - tx^3) = \mathbb{BCT}_0 + (\mathbb{BCT}_1 - r\mathbb{BCT}_0)x + (\mathbb{BCT}_2 - r\mathbb{BCT}_1 - s\mathbb{BCT}_0)x^2.$$

Thus, the proof is completed. \square

Theorem 2. The Binet formula for the bicomplex generalized tribonacci quaternions is:

$$\mathbb{BCT}_n = \frac{A\hat{\alpha}^n}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\hat{\beta}^n}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\hat{\gamma}^n}{(\alpha - \gamma)(\beta - \gamma)}, \quad n \geq 0$$

where $\hat{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3$, $\hat{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3$, and $\hat{\gamma} = \gamma + i\gamma + j\gamma^2 + ij\gamma^3$.

Proof. Using the definition of bicomplex generalized tribonacci quaternion and (3), we have:

$$\begin{aligned} \mathbb{BCT}_n = & \left(\frac{A\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\beta^n}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\gamma^n}{(\alpha - \gamma)(\beta - \gamma)} \right) \\ & + i \left(\frac{A\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\beta^{n+1}}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\gamma^{n+1}}{(\alpha - \gamma)(\beta - \gamma)} \right) \\ & + j \left(\frac{A\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\beta^{n+2}}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\gamma^{n+2}}{(\alpha - \gamma)(\beta - \gamma)} \right) \\ & + ij \left(\frac{A\alpha^{n+3}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\beta^{n+3}}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\gamma^{n+3}}{(\alpha - \gamma)(\beta - \gamma)} \right) \end{aligned}$$

After some basic calculation, the proof is completed. \square

Theorem 3. For $n \in \mathbb{N}$, the exponential generating function for the bicomplex generalized tribonacci quaternions is:

$$\sum_{n=0}^{\infty} \mathbb{BCT}_n \frac{x^n}{n!} = \frac{A\hat{\alpha}e^{\alpha x}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\hat{\beta}e^{\beta x}}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\hat{\gamma}e^{\gamma x}}{(\alpha - \gamma)(\beta - \gamma)},$$

where $\hat{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3$, $\hat{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3$, and $\hat{\gamma} = \gamma + i\gamma + j\gamma^2 + ij\gamma^3$.

Proof. By using the Binet formula for the bicomplex generalized tribonacci quaternions given in Theorem 2, we have:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathbb{BCT}_n \frac{x^n}{n!} &= \frac{A\hat{\alpha}}{(\alpha - \beta)(\alpha - \gamma)} \sum_{n=0}^{\infty} \frac{(\alpha x)^n}{n!} - \frac{B\hat{\beta}}{(\alpha - \beta)(\beta - \gamma)} \sum_{n=0}^{\infty} \frac{(\beta x)^n}{n!} \\ &\quad + \frac{C\hat{\gamma}}{(\alpha - \gamma)(\beta - \gamma)} \sum_{n=0}^{\infty} \frac{(\gamma x)^n}{n!} \\ &= \frac{A\hat{\alpha}e^{\alpha x}}{(\alpha - \beta)(\alpha - \gamma)} - \frac{B\hat{\beta}e^{\beta x}}{(\alpha - \beta)(\beta - \gamma)} + \frac{C\hat{\gamma}e^{\gamma x}}{(\alpha - \gamma)(\beta - \gamma)}. \end{aligned}$$

Thus, the proof is completed. \square

Lemma 1. [9] (Page 239, Lemma 2.3) For the generalized tribonacci numbers, the following equation holds for:

$$\sum_{l=0}^n T_l = \frac{1}{\delta(r, s, t)} (T_{n+2} + (1 - r)T_{n+1} + tT_n + \lambda(r, s, t)), \tag{7}$$

where $\delta(r, s, t) = r + s + t - 1$ and $\lambda(r, s, t) = (r + s - 1)a + (r - 1)b - c$.

Theorem 4. The summation formula for the bicomplex generalized tribonacci quaternions is given by the following:

$$\sum_{l=0}^n \mathbb{BCT}_l = \frac{1}{\delta(r, s, t)} [\mathbb{BCT}_{n+2} + (1 - r)\mathbb{BCT}_{n+1} + t\mathbb{BCT}_n + \lambda(r, s, t)].$$

Proof. Using (4), we have:

$$\sum_{l=0}^n \mathbb{BCT}_l = \sum_{l=0}^n T_l + i \sum_{l=0}^n T_{l+1} + j \sum_{l=0}^n T_{l+2} + ij \sum_{l=0}^n T_{l+3}.$$

Using the Lemma 1, the proof is completed. \square

3. An Application of Bicomplex Generalized Tribonacci Quaternions in Matrices

In this section, we give another method to obtain the n^{th} term of the bicomplex generalized tribonacci quaternion sequence as the calculation of the determinant of a special matrix. For this purpose, we give the following theorem.

Theorem 5. Let $\{x_n\}$ be any third-order linear sequence, defined recursively as:

$$x_{n+1} = rx_n + sx_{n-1} + tx_{n-2}, \quad n \geq 2$$

with the initial values $x_0 = a$, $x_1 = b$, and $x_2 = c$. Then, for all $n \geq 0$,

$$x_n = \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & r & s & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & s \end{vmatrix}_{(n+1) \times (n+1)}$$

Proof. For the proof, we use the induction method on n . The equality holds for $n = 0, 1, 2$. Namely,

$$\begin{aligned} x_0 &= \begin{vmatrix} a \end{vmatrix}_{1 \times 1} = a, \\ x_1 &= \begin{vmatrix} a & -1 \\ b & 0 \end{vmatrix}_{2 \times 2} = b, \\ x_2 &= \begin{vmatrix} a & -1 & 0 \\ b & 0 & -1 \\ c & 0 & 0 \end{vmatrix}_{3 \times 3} = c. \end{aligned}$$

Now, suppose that the equality is true for $2 < k \leq n$. Then, we can verify it for $n + 1$ as follows:

$$\begin{aligned} x_{n+1} &= \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & t & r & s & -1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & s & -1 \\ 0 & 0 & 0 & 0 & 0 & & t & r & s \end{vmatrix}_{(n+2) \times (n+2)} \\ &= s (-1)^{(n+2)+(n+2)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & r & s & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & s \end{vmatrix}_{(n+1) \times (n+1)} \\ &\quad + (-1) (-1)^{(n+2)+(n+1)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & r & s & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & -1 \\ 0 & 0 & 0 & 0 & 0 & & t & r \end{vmatrix}_{(n+1) \times (n+1)} \end{aligned}$$

$$\begin{aligned}
 x_{n+1} &= s x_n + r \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & t & r & s & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s \end{vmatrix}_{n \times n} + \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & t & r & s & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & t \end{vmatrix}_{n \times n} \\
 &= s x_n + r x_{n-1} + \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & t & r & s & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & t \end{vmatrix}_{n \times n} \\
 &= s x_n + r x_{n-1} + t (-1)^{n+n} \begin{vmatrix} a & -1 & 0 & 0 & 0 & \dots & 0 \\ b & 0 & -1 & 0 & 0 & \dots & 0 \\ c & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & t & r & s & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 \end{vmatrix}_{(n-1) \times (n-1)} \\
 &= s x_n + r x_{n-1} + t x_{n-2}.
 \end{aligned}$$

Thus, the proof is completed. \square

Using the above theorem, we have the following result, which gives a different way to obtain the n^{th} term of bicomplex generalized tribonacci quaternions.

Theorem 6. For $n \geq 0$, we have:

$$\mathbb{BCT}_n = \begin{vmatrix} \mathbb{BCT}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathbb{BCT}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathbb{BCT}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & r & s & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & s \end{vmatrix}_{(n+1) \times (n+1)}.$$

Proof. By using (5) and the previous theorem, the proof can be easily shown. \square

4. Conclusions

In this paper, we defined the bicomplex generalized tribonacci quaternions. Some properties involving this sequence, including the Binet formula, generating function, exponential generating function, and the summation formula, were presented. Moreover, a special matrix whose entries are bicomplex generalized tribonacci quaternions was presented as a different way to obtain the n^{th} term of the bicomplex generalized tribonacci quaternions sequence. One of the aims of this work is to obtain bicomplex quaternion versions of tribonacci or tribonacci-like sequences in the literature.

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