The Ramanujan Journal (2021) 54:555–569
https://doi.org/10.1007/s11139-019-00210-8

Variations around a general quantum operator

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Received: 7 May 2019 / Accepted: 26 August 2019 / Published online: 17 February 2020

Abstract
Let $I \subseteq \mathbb{R}$ be an interval and $\beta : I \to I$ a strictly increasing and continuous function with a unique fixed point $s_0 \in I$ that satisfies $(s_0 - t)(\beta(t) - t) \geq 0$ for all $t \in I$, where the equality holds only when $t = s_0$. For appropriate choices of the function β, the quantum operator defined by Hamza et al., $D_\beta[f](t) := \dfrac{f\big(\beta(t)\big) - f(t)}{\beta(t) - t}$ if $t \neq s_0$ and $D_\beta[f](s_0) := f'(s_0)$ if $t = s_0$, generalizes both the Jackson q-operator D_q and the Hahn (quantum derivative) operator, $D_{q,\omega}$. With respect to the inverse of this general quantum difference operator, the β-integral, we study properties of the corresponding Lebesgue spaces $\mathscr{L}_\beta^p([a, b])$.

Keywords General quantum difference operator · β-Derivative · β-Integral · β-Lebesgue spaces · q-Analogues · Jackson q-integral

Mathematics Subject Classification 33E20 · 33E30 · 40A05 · 40A10

1 Introduction

Fixed $0 < q < 1$ and $\omega \geq 0$, the (forward difference) ω-derivative,

$$\triangle_\omega[f](x) := \frac{f(x + \omega) - f(x)}{\omega}$$

and the Jackson q-derivative

$$D_q[f](x) := \frac{f(qx) - f(x)}{(q - 1)x},$$

This research was partially supported by Portuguese Funds through the FCT — Fundação para a Ciência e a Tecnologia—within the Project UID/MAT/00013/2013.

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are particular cases of the (q, ω)-derivative operator (the Hahn's quantum operator),

$$D_{q,\omega}[f](x) := \frac{f\big(qx+\omega\big) - f(x)}{(q-1)x+\omega}, \tag{1}$$

which, in turn, as we shall see, is a particular case of the more general quantum operator D_β described in the abstract.

The importance of these operators can be assessed through the numerous publications and its different approaches and perspectives: far from being exhaustive, for instance, the Hahn quantum variational calculus and the q-difference equations properties [6,7,10,13,32,37] or, new characterizations and new properties related with families of orthogonal q-polynomials or q-exponential families [27,33,38].

Of particular relevance are the corresponding inverse operators, which enable one to define, the following integrals, respectively: the Nörlund integral

$$\int_a^b f\,\Delta_\omega := \omega \sum_{k=0}^{+\infty} \big[f(b+k\omega) - f(a+k\omega)\big],$$

the Jackson q-integral

$$\int_a^b f\,\mathrm{d}_q := (1-q)\sum_{k=0}^{+\infty} \big[bf(bq^k) - f(aq^k)q^k\big]q^k$$

and the Jackson–Thomae–Nörlund (q, ω)-integral

$$\int_a^b f\,\mathrm{d}_{q,\omega} := \int_{\omega_0}^b f\,\mathrm{d}_{q,\omega} - \int_{\omega_0}^a f\,\mathrm{d}_{q,\omega}.$$

For the introduction of this concepts see [29–31] and for more details over the q-integrals see, for example [11].

Regarding or involving these inverse operators one can find a large variety of publications. For instance: properties of the Jackson q-integral for q-commuting variables were studied in Sect. 7 of [35]; q-type sampling theorems with q-versions of the classical sampling theorem of Whittaker, Kotel'nikov and Shannon, where the Fourier transform is replaced by a q-type one defined in terms of Jackson's q-integral [1,8,9,12,26,28]; Paley–Wiener theorems in [3,21,22] by considering q-transforms (see for instance [36]) defined in terms of the Jackson's q-integral; in p. 31, Sect. 6, of [34], a q-analogue of the Weyl fraction integral operator is introduced. It is remarked there that this q-analogue can be represented in terms of an iterated infinite q-integral of Jackson; basic Fourier expansions [5,14–18] where each orthogonality was established via an inner product associated with the Jackson's q-integral. In [19], properties involving the Jackson and the Hahn operators, as well as its inverse operators, respectively, the Jackson q-integral and the Jackson–Thomae–Nörlund (q, ω)-integral were obtained. Regarding these topics and closely related ones see also [2,4].

In 2015, Hamza et al. [24] introduced a general quantum difference operator, the β-derivative, generalizing the Hahn's quantum operator (for certain functions β), and its inverse operator, the β-integral. Also in 2015 [23], β-Hölder, β-Minkowski, β-Gronwall, β-Bernoulli and β-Lyapunov inequalities were proved. Later, in [25], the exponential, trigonometric and hyperbolic functions were introduced and, in [20], a new variational calculus was developed, being both publications based in the above mentioned general quantum difference operator.

The target of the present work is to obtain properties for the β-integral, more precisely, after establishing sufficient conditions in order to guarantee that q-type Hölder and Minkowski inequalities hold, we consider a q-analogue of the Lebesgue function space associated with the β-integral, and we assure that it becomes a Banach space, separable and reflexive. We also introduce the versions of the β-integrals defined over unbounded intervals.

In Sect. 2 we collect the definitions and the results that are required to proceed to Sect. 3, where the outcome of the present work is exhibited. We believe that Theorems 3 and 4 are original. We also point out the definitions of the infinite β-integrals in Remark 1, the Proposition 2 and the less restrictive statements of Theorems 5 and 6.

2 The β-derivative and the β-integral

2.1 The β-derivative

In the following, $I \subseteq \mathbb{R}$ will denote an interval and $\beta : I \to I$ a strictly increasing and continuous function with a unique fixed point $s_0 \in I$ satisfying

$$(t - s_0)(\beta(t) - t) \leq 0 \tag{2}$$

for all $t \in I$, where the equality holds only when $t = s_0$.

For functions $f : I \to \mathbb{K}$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$ [1], Hamza et al. [24] defined the general quantum difference operator

$$D_\beta[f](t) := \begin{cases} \dfrac{f\big(\beta(t)\big) - f(t)}{\beta(t) - t} & \text{if } \ t \neq s_0, \\ f'(s_0) & \text{if } \ t = s_0. \end{cases} \tag{3}$$

provided that $f'(s_0)$ exists. $D_\beta[f](t)$ is called the β-derivative of f at $t \in I$. One says that f is β-differentiable on I if $f'(s_0)$ exists.

It is clear that the Hahn operator (1) corresponds to the choice $\beta(t) = qt + \omega$ being the fixed point given by $s_0 = \frac{\omega}{1-q}$.

Accordingly to [24], one can replace condition (2) by $(t - s_0)(\beta(t) - t) \geq 0$ for $t \in I$.

[1] In fact, $\mathbb{K} = \mathbb{X}$ can represent any Banach space [24, p.2]

2.2 The β-integral

Consider the notation $\beta^k(t) := (\underbrace{\beta \circ \beta \circ \ldots \circ \beta}_{k\,\text{times}})(t)$ and $\beta^0(t) := t$, for $t \in I$ and $k = 1, 2, 3, \cdots$, as well as the β-interval with extreme points a and b,

$$[a, b]_\beta := \left\{ \beta^n(x) \mid (x, n) \in \{a, b\} \times \mathbb{N}_0 \right\}.$$

Clearly, for every real numbers a and b, the following property holds:

$$a, b \in I \quad \Rightarrow \quad [a, b]_\beta \subset I.$$

We state here the following proposition:

Proposition 1 [24, Lemma 2.1, p. 3] *The sequence of functions* $\left\{\beta^k(t)\right\}_{k\in\mathbb{N}_0}$ *converges uniformly to the constant function* $\hat{\beta}(t) := s_0$ *on every compact interval* $J \subset I$ *containing* s_0.

The quantum difference inverse operator, the β-integral, with $a, b \in I$, is defined by

$$\int_a^b f \, \mathrm{d}_\beta := \int_{s_0}^b f \, \mathrm{d}_\beta - \int_{s_0}^a f \, \mathrm{d}_\beta, \tag{4}$$

where

$$\int_{s_0}^x f \, \mathrm{d}_\beta := \sum_{k=0}^{+\infty} \left(\beta^k(x) - \beta^{k+1}(x)\right) f\left(\beta^k(x)\right). \tag{5}$$

Thus,

$$\int_a^b f \, \mathrm{d}_\beta = \sum_{k=0}^{+\infty} \left(\beta^k(b) - \beta^{k+1}(b)\right) f\left(\beta^k(b)\right) - \sum_{k=0}^{+\infty} \left(\beta^k(a) - \beta^{k+1}(a)\right) f\left(\beta^k(a)\right). \tag{6}$$

If the infinite sum in the right side of (5) is convergent then we say that the function f is β-integrable in $[s_0, x]$. The β-integral in the left side of (4) is well defined provided that at least one of the β-integrals in the right side is finite and we say that f is β-integrable in $[a, b]$ if it is both β-integrable in $[s_0, a]$ and in $[s_0, b]$.
When $\beta(t) = qt + \omega$ with $0 < q < 1$ and $\omega \geq 0$, the resulting q-integral is precisely the Jackson–Thomae–Nörlund integral, where the corresponding fixed point is given by $s_0 = \omega_0 = \frac{\omega}{1-q}$; if either $\omega = 0$ then one obtains the Jackson q-integral, being $s_0 = 0$ the fixed point of the relative β function.

Remark 1 If $1 \in I$, $\beta(I) \subset I$, $\beta^{-1}(I) \subset I$ and $s_0 < 1$, it is possible to consider the infinite β-integral

$$\int_{s_0}^{+\infty} f \, \mathrm{d}_\beta := \sum_{k=-\infty}^{+\infty} \Big(\beta^k(1) - \beta^{k+1}(1)\Big) f\big(\beta^k(1)\big) \tag{7}$$

and, in a similar way, if $\pm 1 \in I$, $\pm\beta(I) \subset I$, $\pm\beta^{-1}(I) \subset I$ and $-1 < s_0 < 1$ one can define

$$\int_{-\infty}^{+\infty} f \, \mathrm{d}_\beta := \sum_{k=-\infty}^{+\infty} \Big(\beta^k(1) - \beta^{k+1}(1)\Big)\Big[f\big(\beta^k(1)\big) + f\big(-\beta^k(1)\big)\Big], \tag{8}$$

whenever the corresponding series converges. Here, for a positive integer k, we take $\beta^{-k}(t) := (\underbrace{\beta^{-1} \circ \beta^{-1} \circ \ldots \circ \beta^{-1}}_{k \text{ times}})(t)$.

If $\beta(t) = qt$, $0 < q < 1$, then we recover the infinite Jackson's q-integrals $\int_0^{+\infty} f \, \mathrm{d}_q := (1-q)\sum_{n=-\infty}^{+\infty} f(q^n)q^n$ and $\int_{-\infty}^{+\infty} f \, \mathrm{d}_q := (1-q)\sum_{n=-\infty}^{+\infty}[f(q^n) + f(-q^n)]q^n$.

We notice that, under appropriate conditions, the properties that will be presented in the following for the β-integral in $[a, b]$ may be extended to the infinite β-integrals (7) and (8).

3 The spaces $\mathscr{L}_\beta^p[a, b]$ and $L_\beta^p[a, b]$

The β-integral in $[a, b]$ verifies a list of properties ([24, Lemma 3.5, p. 13]) such as the *linearity* whereas

$$\int_a^b (f+g) \, \mathrm{d}_\beta = \int_a^b f \, \mathrm{d}_\beta + \int_a^b g \, \mathrm{d}_\beta$$

and, for any fixed k,

$$\int_a^b (kf) \, \mathrm{d}_\beta = k \int_a^b f \, \mathrm{d}_\beta.$$

Since $\int_a^b f \, \mathrm{d}_\beta = -\int_b^a f \, \mathrm{d}_\beta$ and $\int_a^a f \, \mathrm{d}_\beta = 0$ we can admit in the definition of the β-integral (4) that $a < b$. Using identity (6) and the linearity properties, one can prove the following important *monotony* property for the β-integral.

Proposition 2 *If s_0 is the fixed point of β and a and b are elements of I such that $a \leq s_0 \leq b$, then*

$$f \leq g \ \text{in} \ [a, b]_\beta \implies \int_a^b f \, \mathrm{d}_\beta \leq \int_a^b g \, \mathrm{d}_\beta. \tag{9}$$

In particular, $\int_a^b f\,\mathrm{d}_\beta \geq 0$ *whenever* $f \geq 0$ *in* $[a,b]_\beta$.

This important property (9) is crucial to prove the Hölder inequality involving the β-integral.

3.1 The space $\mathscr{L}_\beta^p[a,b]$

For $a, b \in I$, we will denote by $\mathscr{L}_\beta^p[a,b]$ the set of functions $f : I \to \mathbb{C}$ such that $|f|^p$ is β-integrable in $[a,b]$, i.e.,

$$\mathscr{L}_\beta^p[a,b] = \left\{ f : I \to \mathbb{C} \;\middle|\; \int_a^b |f|^p \mathrm{d}_\beta < \infty \right\}.$$

We also set

$$\mathscr{L}_\beta^\infty[a,b] = \left\{ f : I \to \mathbb{C} \;\middle|\; \sup_{k\in\mathbb{N}_0} \left\{ \left|f\left(\beta^k(a)\right)\right|, \left|f\left(\beta^k(b)\right)\right| \right\} < \infty \right\}.$$

The next theorem shows that a β-type Hölder inequality holds, provided we take $a \leq s_0 \leq b$ and $p > 1$. As usual, by p' we denote the conjugate exponent of a real number $p \geq 1$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention $p' = \infty$ if $p = 1$. Its proof will be omitted since, by (9), it can be carried out using an argumentation similar to the one used to prove the corresponding classical Hölder inequality for the Riemann or Lebesgue integrals. A similar proof can be found in [23].

Theorem 1 *If* $a \leq s_0 \leq b$ *and* $1 < p < \infty$, *then*

$$\int_a^b |fg| \,\mathrm{d}_\beta \leq \left(\int_a^b |f|^p \,\mathrm{d}_\beta \right)^{\frac{1}{p}} \left(\int_a^b |g|^{p'} \,\mathrm{d}_\beta \right)^{\frac{1}{p'}}, \tag{10}$$

whenever $f \in \mathscr{L}_\beta^p[a,b]$ *and* $g \in \mathscr{L}_\beta^{p'}[a,b]$.

Remark 2 When $p = 1$ and $a \leq s_0 \leq b$, it follows immediately from (4)–(5) and (9) that the inequality

$$\int_a^b |fg| \,\mathrm{d}_\beta \leq \sup_{k\in\mathbb{N}_0} \left\{ \left|g\left(\beta^k(a)\right)\right|, \left|g\left(\beta^k(b)\right)\right| \right\} \int_a^b |f| \,\mathrm{d}_\beta$$

holds, provided $f \in \mathscr{L}_\beta^1[a,b]$ and $g \in \mathscr{L}_\beta^\infty[a,b]$.

The following theorem is a β-type version of the Minkowski's inequality and is an important outcome of (10).

Theorem 2 *If* $a \leq s_0 \leq b$ *and* $1 \leq p < \infty$, *then*

$$\left(\int_a^b |f+g|^p \,\mathrm{d}_\beta \right)^{\frac{1}{p}} \leq \left(\int_a^b |f|^p \,\mathrm{d}_\beta \right)^{\frac{1}{p}} + \left(\int_a^b |g|^p \,\mathrm{d}_\beta \right)^{\frac{1}{p}}$$

for all $f, g \in \mathscr{L}_\beta^p[a,b]$.

As an immediate consequence of the β-Minkowski inequality we may state the following important property (the case $p=\infty$ is trivial).

Corollary 1 *If* $a \le s_0 \le b$ *and* $1 \le p \le \infty$*, then the set* $\mathscr{L}_\beta^p[a,b]$*, with the usual operations of addition of functions and multiplication of a function by a number (real or complex), becomes a linear space over* $\mathbb{K}$.

3.2 The space $L_\beta^p[a,b]$

For $f, g \in \mathscr{L}_\beta^p[a,b]$, we write $f \sim g$ if

$$f\big(\beta^k(a)\big) = g\big(\beta^k(a)\big) \quad \text{and} \quad f\big(\beta^k(b)\big) = g\big(\beta^k(b)\big) \tag{11}$$

holds for all $k = 0, 1, 2, \ldots$, i.e., we say that $f \sim g$ if $f = g$ in $[a,b]_\beta$. Clearly, $\sim$ defines an equivalence relation in $\mathscr{L}_\beta^p[a,b]$. We will represent by $L_\beta^p[a,b]$ the corresponding quotient set:

$$L_\beta^p[a,b] := \mathscr{L}_\beta^p[a,b] \big/ \sim \; .$$

Theorem 3 *If* $a \le s_0 \le b$ *and* $1 \le p \le \infty$ *then* $L_\beta^p[a,b]$ *is a normed linear space over* $\mathbb{K}$ *with norm*

$$\|f\|_{L_\beta^p[a,b]} := \begin{cases} \left(\displaystyle\int_a^b |f|^p \, \mathrm{d}_\beta \right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty, \\ \displaystyle\sup_{k \in \mathbb{N}_0} \left\{ \left| f\big(\beta^k(a)\big) \right|, \left| f\big(\beta^k(b)\big) \right| \right\} & \text{if} \quad p = \infty. \end{cases} \tag{12}$$

Remark 3 As usual, on the right-hand side of (12), f denotes any representative (i.e., a function in $\mathscr{L}_\beta^p[a,b]$) of the class $f \in L_\beta^p[a,b]$ appearing in the norm on the left-hand side. Of course, in view of (6) and (11), the definition of the norm $\|f\|_{L_\beta^p[a,b]}$ is independent of the chosen representative.

Proof – Case $1 \le p < \infty$.

The triangle inequality is precisely the β-Minkowski's inequality (Theorem 2), while

$$\|\lambda f\|_{L_\beta^p[a,b]} = |\lambda| \, \|f\|_{L_\beta^p[a,b]}$$

holds trivially for every $\lambda \in \mathbb{R}(\mathbb{C})$ and every $f \in L_\beta^p[a,b]$.

On one hand, by the definition (12) and by (9), we have

$$\|f\|_{L_\beta^p[a,b]} = 0 \implies f = 0 \text{ in } L_\beta^p[a,b].$$

In fact, s_0 is the unique fixed point of β, this function is strictly increasing in I and satisfies the condition (2) hence, it follows immediately

$$s_0 < \beta^{k+1}(x) < \beta^k(x) \quad \text{whenever} \quad x > s_0 \tag{13}$$

and[2]

$$\beta^k(x) < \beta^{k+1}(x) < s_0 \quad \text{whenever} \quad x < s_0. \tag{14}$$

By (12) and (6) the value of $\|f\|^p_{L^p_\beta[a,b]}$ is given by

$$\sum_{k=0}^{+\infty}\Big(\beta^k(b)-\beta^{k+1}(b)\Big)|f\big(\beta^k(b)\big)|^p-\sum_{k=0}^{+\infty}\Big(\beta^k(a)-\beta^{k+1}(a)\Big)|f\big(\beta^k(a)\big)|^p, \tag{15}$$

thus, when $a < s_0 < b$, by (13) and (14), we have that $\|f\|_{L^p_\beta[a,b]} = 0$ implies necessarily

$$f\big(\beta^k(a)\big)=0,\ k=0,1,2,\ldots \quad \text{and} \quad f\big(\beta^k(b)\big)=0,\ k=0,1,2,\ldots,$$

which proves that

$$\|f\|_{L^p_\beta[a,b]}=0 \implies f=0 \text{ in } [a,b]_\beta.$$

– Case $p = \infty$.
  In this case it is trivial to check the axioms of a norm.

Therefore, (12) indeed defines a norm in $L^p_\beta[a,b]$ for $1 \le p \le \infty$. □

The following theorem generalizes Theorem 3.8 of [19, p. 347].

Theorem 4 *If $a \le s_0 \le b$ and $1 \le p \le \infty$, then the following holds:*

(i) $L^p_\beta[a,b]$, *endowed with the norm* (12), *is a Banach space for* $1 \le p \le \infty$, *which is separable if* $1 \le p < \infty$ *and reflexive if* $1 < p < \infty$.
(ii) $L^2_\beta[a,b]$ *is a Hilbert space with inner product*

$$\langle f,g\rangle_\beta := \int_a^b f\,\overline{g}\,\mathrm{d}_\beta, \quad f,g \in L^2_\beta[a,b]. \tag{16}$$

Proof During the proof we will fix $a, b \in I$.
(i) By Theorem 3, $L^p_\beta[a,b]$ is a normed space.

[2] This shows that, for every fixed $x \in I$, the sequence $\{\beta^k(x)\}_k$ is strictly monotone decreasing or strictly monotone increasing according to $x > s_0$ or $x < s_0$, respectively. Proposition 1 shows that it converges in both cases to the fix point s_0.

– Let one assume that $1 \leq p < \infty$ and consider the space $\ell^p_{\beta,a}$ of all sequences $x = (\xi_n)_n$ such that

$$\sum_{k=0}^{+\infty} \Big(\beta^{k+1}(a) - \beta^k(a)\Big)|\xi_k|^p < \infty. \tag{17}$$

Then, gifted with the norm

$$\|x\|_{\ell^p_{\beta,a}} := \left(\sum_{k=0}^{+\infty} \Big(\beta^{k+1}(a) - \beta^k(a)\Big)|\xi_k|^p\right)^{\frac{1}{p}},$$

$\ell^p_{\beta,a}$ becomes a Banach space if $1 \leq p < \infty$, separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$. This is an outcome of the following event: for any sequence $x = (\xi_n)_n$, we have $x \in \ell^p_{\beta,a}$ if and only if $x_{\beta,a} \in \ell^p$, where $x_{\beta,a} := ((\beta^{n+1}(a) - \beta^n(a))^{\frac{1}{p}}\xi_n)_n$, and, in such a case, the equality

$$\|x\|_{\ell^p_{\beta,a}} = \|x_{\beta,a}\|_{\ell^p}$$

holds. In fact, the mapping $U_{\beta,a} : \ell^p_{\beta,a} \to \ell^p$ $(x \mapsto x_{\beta,a})$ is an isometric isomorphism.
Arguing identically one can show that the space $\ell^p_{\beta,b}$ of all sequences $y = (\zeta_n)_n$ such that

$$\sum_{k=0}^{+\infty} \Big(\beta^k(b) - \beta^{k+1}(b)\Big)|\zeta_k|^p < \infty,$$

endowed with the norm

$$\|x\|_{\ell^p_{\beta,b}} := \left(\sum_{k=0}^{+\infty} \Big(\beta^k(b) - \beta^{k+1}(b)\Big)|\xi_k|^p\right)^{\frac{1}{p}},$$

is also a Banach space if $1 \leq p < \infty$, separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$. In fact, for any sequence, $x = (\zeta_n)_n$, we have $x \in \ell^p_{\beta,b}$ if and only if $x_{\beta,b} \in \ell^p$, where $x_{\beta,b} := ((\beta^n(b) - \beta^{n+1}(b))^{\frac{1}{p}}\xi_n)_n$, and, in such a case, the equality

$$\|x\|_{\ell^p_{\beta,b}} = \|x_{\beta,b}\|_{\ell^p}$$

holds. This is true because the mapping $U_{\beta,b} : \ell^p_{\beta,b} \to \ell^p$ $(x \mapsto x_{\beta,b})$ is an isometric isomorphism.

Furthermore, taking into account (4), (5), (12) and (15), we see that

$$f \in L^p_\beta[a,b] \quad \text{if and only if} \quad \big(f(\beta^n(a)\big)_n \in \ell^p_{\beta,a} \;\wedge\; \big(f(\beta^n(b)\big)_n \in \ell^p_{\beta,b}.$$

Therefrom, the equality

$$\|f\|_{L^p_\beta[a,b]} = \left(\left\| \Big(f\big(\beta^n(b)\big)\Big)_n \right\|^p_{\ell^p_{\beta,b}} + \left\| \Big(f\big(\beta^n(a)\big)\Big)_n \right\|^p_{\ell^p_{\beta,a}} \right)^{1/p} \tag{18}$$

holds.
Now, consider the product space $\ell^p_{\beta,a} \times \ell^p_{\beta,b}$ endowed with the norm

$$\|(x,y)\|_{\ell^p_{\beta,a}\times\ell^p_{\beta,b}} := \left(\|y\|^p_{\ell^p_{\beta,b}} + \|x\|^p_{\ell^p_{\beta,a}} \right)^{1/p}.$$

Then $\ell^p_{\beta,a} \times \ell^p_{\beta,b}$ is a Banach space if $1 \le p < \infty$, separable if $1 \le p < \infty$ and reflexive if $1 < p < \infty$. In addition, the mapping $S : L^p_\beta[a,b] \to \ell^p_{\beta,a} \times \ell^p_{\beta,b}$ defined by

$$Sf := \left(\Big(f\big(\beta^n(a)\big)\Big)_n, \Big(f\big(\beta^n(b)\big)\Big)_n \right)$$

is an isometric isomorphism. As a matter of fact, S is linear and, by (18), it is an isometry. To prove that S is onto, take any pair of sequences $(x,y) \in \ell^p_{\beta,a} \times \ell^p_{\beta,b}$. Set $x = (\xi_n)_n$ and $y = (\zeta_n)_n$ and take $f : I \to \mathbb{C}$ such that $f|[a,b]_\beta$ is defined by

$$f(t) := \begin{cases} \xi_n, & \text{if} \quad t = \beta^n(a), \quad n = 0,1,2,\ldots \\ \zeta_n, & \text{if} \quad t = \beta^n(b), \quad n = 0,1,2,\ldots. \end{cases}$$

(It does not matter how one defines the function $f(t)$ for $t \neq \beta^n(a)$ and $t \neq \beta^n(b)$.) Since $x = (f(\beta^n(a)))_n \in \ell^p_{\beta,a}$ and $y = (f(\beta^n(b)))_n \in \ell^p_{\beta,b}$ then, from (18), we derive that $f \in L^p_\beta[a,b]$, and, of course, $Sf = (x,y)$. This completes the proof of (i) for $1 \le p < \infty$.

– The proof for $p = \infty$ is now easy.

(ii) Its proof is a consequence of the following fact: the complex function $\langle .,. \rangle_\beta$ defined in $L^2_\beta[a,b] \times L^2_\beta[a,b]$ by $\langle f, g\rangle_\beta := \int_a^b f\,\overline{g}\,\mathrm{d}_\beta$ satisfies the axioms of an inner product space. For instance, for any $f \in L^2_\beta[a,b]$,

$$\begin{aligned}\langle f, f\rangle_\beta &= \|f\|_\beta^2 = \int_a^b |f|^2 \,\mathrm{d}_\beta \\ &= \sum_{k=0}^{+\infty}\Big(\beta^k(b)-\beta^{k+1}(b)\Big)|f\big(\beta^k(b)\big)|^2 \\ &\quad - \sum_{k=0}^{+\infty}\Big(\beta^k(a)-\beta^{k+1}(a)\Big)|f\big(\beta^k(a)\big)|^2 \geq 0\end{aligned}$$

being the last nonnegativity guaranteed by the assumption $a \leq s_0 \leq b$ along with (13) and (14). Also, with the same argumentation, if $f \neq 0$ in $[a,b]_\beta$ then $\langle f, f\rangle_\beta = \|f\|^2 = \int_a^b |f| \,\mathrm{d}_\beta > 0$. The remaining axioms of an inner product space are trivial to verify which shows, together with part (i) for $p = 2$, that $L^2_\beta[a,b]$ is a Hilbert space with respect to the inner-product (16). □

3.3 Properties of the β-derivative and of the β-integral

We go back to the introduction of Sect. 2, where the definition of the β-derivative operator (3) was introduced in Sect. 2.1:

$$D_\beta[f](t) := \begin{cases} \dfrac{f\big(\beta(t)\big)-f(t)}{\beta(t)-t} & \text{if} \quad t \neq s_0, \\ f'(s_0) & \text{if} \quad t = s_0. \end{cases}$$

Notice that if f is differentiable at a point $t \in I$, then

$$\lim_{\beta(t)\to t} D_\beta[f](t) = f'(t),$$

hence D_β is a $beta$-analogue of the standard derivative operator.
The β-derivative satisfies properties which may be regarded as β-analogues of the corresponding properties for the usual derivative. For instance, the quantum operator (3) is linear, i.e.,

$$D_\beta[\alpha f + \beta g](t) = \alpha D_\beta[f](t) + \beta D_\beta[g](t),$$

where α and β are any real or complex numbers, and satisfies the following β-product rule: for $t \in I$,

$$\begin{aligned}D_\beta[f\cdot g](t) &= D_\beta[f](t)\cdot g(t) + f\big(\beta(t)\big)\cdot D_\beta[g](t) \\ &= D_\beta[g](t)\cdot f(t) + g\big(\beta(t)\big)\cdot D_\beta[f](t)\end{aligned} \tag{19}$$

if f and g are β-differentiable in I. Also, f will be the constant function such that $f(t) = f(s_0)$ for all $t \in I$ whenever $D_\beta[f](t) = 0$ for all $t \in I$. For these and other properties of the general quantum difference operator D_β see [20,24]. These equalities hold for all $t \neq s_0$, and also for $t = s_0$ whenever $f'(s_0)$ and $g'(s_0)$ exist.

3.4 The fundamental theorem of β-calculus

The next proposition states a β-analogue of the fundamental theorem of calculus for the Riemann integral.

Theorem 5 *Let $\beta : I \to I$ be a function satisfying the conditions described in Sect. 2.1. Fix $a, b \in I$ and let $f : I \to \mathbb{K}$ be a function such that $D_\beta[f] \in \mathscr{L}^1_\beta[a,b]$. Then:*

(i) *The equality*

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta = \left[f(s) - \lim_{k\to+\infty} f\big(\beta^k(s)\big) \right]_{s=a}^b$$

holds, provided the involved limits exist.

(ii) *In addition, assuming that $a < s_0 < b$, if f has a discontinuity of first kind at s_0 then*

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta = f(b) - f(a) - \Big(f(s_0^+) - f(s_0^-) \Big).$$

Of course, if f is continuous at s_0 then

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta = f(b) - f(a).$$

Proof (i) From (6), we may write

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta$$
$$= \sum_{k=0}^{+\infty} \Big[\Big(\beta^k(b) - \beta^{k+1}(b)\Big) D_\beta[f]\big(\beta^k(b)\big) - \Big(\beta^k(a) - \beta^{k+1}(a)\Big) D_\beta[f]\big(\beta^k(a)\big) \Big].$$

Now, using the definition of the β-derivative (3) and simplifying, one obtains

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta = \sum_{k=0}^{+\infty} \Big[f\big(\beta^k(b)\big) - f\big(\beta^{k+1}(b)\big) - \Big(f\big(\beta^k(a)\big) - f\big(\beta^{k+1}(a)\big) \Big) \Big].$$

Telescoping the infinite sum of the right side it results

$$\int_a^b D_\beta[f]\,\mathrm{d}_\beta = f(b) - \lim_{k\to+\infty} f\big(\beta^{k+1}(b)\big) - \Big(f(a) - \lim_{k\to+\infty} f\big(\beta^{k+1}(a)\big) \Big), \tag{20}$$

and this proves (i).

(ii) Using (13) and (14) together with the hypothesis $a < s_0 < b$, and introducing it in (20) the result follows. □

3.5 The β-integration by parts formula

Now we state the β-analogue of the integration by parts formula.

Theorem 6 *Let $\beta : I \to I$ be a function satisfying the conditions described in Sect. 2.1. Fix $a, b \in I$ and two functions $f : I \to \mathbb{K}$ and $g : I \to \mathbb{K}$. Then:*

$$\int_a^b f \cdot D_\beta[g]\,\mathrm{d}_\beta = \Big[(f \cdot g)(s) - \lim_{k\to+\infty}(f \cdot g)\big(\beta^k(s)\big)\Big]_{s=a}^b - \int_a^b \big(g \circ \beta\big) \cdot D_\beta[f]\,\mathrm{d}_\beta$$

holds, provided $f, g \in \mathscr{L}_\beta^1[a,b]$, $D_\beta[f]$ and $D_\beta[g]$ are bounded in $[a,b]_\beta$, and the limits exist.

If, in addition, f and g are continuous at s_0 then

$$\int_a^b f \cdot D_\beta[g]\,\mathrm{d}_\beta = \Big[f \cdot g\Big]_a^b - \int_a^b \big(g \circ \beta\big) \cdot D_\beta[f]\,\mathrm{d}_\beta.$$

Proof By the β-product rule (19) one has

$$f(t)D_\beta[g](t) = D_\beta[f \cdot g](t) - g\big(\beta(t)\big) \cdot D_\beta[t](t),$$

therefore, β-integrating both sides of this equality over the interval $[a, b]$ and taking into account Theorem 5, the result follows. (Notice that the above equality and the hypothesis on f and g gives $D_\beta[f \cdot g] = f \cdot D_\beta[g] + \big(g \circ \beta\big) \cdot D_q[f] \in \mathscr{L}_q^1[a,b]$, hence the assumptions of Theorem 5 are fulfilled.) □

Acknowledgements The author is grateful to J. Petronilho for the interesting discussions.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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