



SOME FURTHER NOTES ON THE MATRIX EQUATIONS $A^T X B + B^T X^T A = C$ AND $A^T X B + B^T X A = C^*$



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Abstract Dehghan and Hajarian, [4], investigated the matrix equations $A^T X B + B^T X^T A = C$ and $A^T X B + B^T X A = C$ providing inequalities for the determinant of the solutions of these equations. In the same paper, the authors presented a lower bound for the product of the eigenvalues of the solutions to these matrix equations. Inspired by their work, we give some generalizations of Dehghan and Hajarian results. Using the theory of the numerical ranges, we present an inequality involving the trace of C when A, B, X are normal matrices satisfying $A^T B = B A^T$.

Key words matrix equation; eigenvalue; trace; permutation matrix

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1 Introduction

We denote by M_n the set of all $n \times n$ real matrices. For $A \in M_n$, we denote by A^T , $\text{Tr}(A)$ and $\det(A)$, the transpose, the trace and the determinant of A , respectively. The spectrum of $A \in M_n$ will be represented by $\rho(A)$. For a matrix $A \in M_n$, with spectrum $\rho(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, the determinant of A is the product of all the eigenvalues: $\lambda_1 \lambda_2 \cdots \lambda_n$ and the trace of A is the sum of all the eigenvalues: $\lambda_1 + \cdots + \lambda_n$. We say that A is a real stability matrix if $\text{Re } \lambda_i < 0$, for $i = 1, \dots, n$.

An active research is being conducted around the topic of matrix equations, since is widely used in many different areas such as computational mathematics. The research groups that work on this topic are interested for instance in obtaining several bounds for the eigenvalues, the trace and the determinant of the solutions of some matrix equations (e.g. Lyapunov and Riccati equation) [5, 6]. The labor around this topic can also be seen searching in MathScinet database for the expression “matrix equation”, which will return over 1600 items.

In this paper, we focus on the study of the following matrix equations

$$A^T X B + B^T X^T A = C, \quad (1.1)$$

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and

$$A^T X B + B^T X A = C, \quad (1.2)$$

where $A, B, C \in M_n$ are real known matrices and $X \in M_n$ is an unknown real matrix. The paper is organized as follows: in Section 2, we obtain generalizations of Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 of [4]. In Section 3, considering in (1.1) and (1.2), C a $n \times n$ matrix, we obtain an inequality involving the trace of C and the eigenvalues of the normal matrices A, B, X satisfying $A^T B = B A^T$.

2 Inequalities for the Determinant

In this section, we revisit the results in [4] giving different proofs. First, we present a theorem due to Ostrowski and Taussky [2] which is an essential tool to prove the main result of this section.

Theorem 2.1 If $A \in M_n$ is such that $\frac{A+A^T}{2}$ is positive definite, then $\det\left(\frac{A+A^T}{2}\right) \leq \det A$. Equality holds if and only if A is symmetric.

Next, we present a different proof of Theorem 2.1 [4] using the above theorem.

Theorem 2.2 Let $A, B, X \in M_n$ and let C be a $n \times n$ positive definite matrix. If the matrix equation (1.1) is consistent, then

$$\det(C) \leq 2^n \det(A) \det(X) \det(B). \quad (2.1)$$

The equality holds if and only if $C = 2 A^T X B$.

Proof Equation (1.1) is equivalent to

$$\frac{A^T X B + (A^T X B)^T}{2} = \frac{C}{2}. \quad (2.2)$$

Since C is a positive definite matrix, by Theorem 2.1, we have

$$\det\left(\frac{A^T X B + (A^T X B)^T}{2}\right) \leq \det(A^T X B), \quad (2.3)$$

which is equivalent to $\det\left(\frac{C}{2}\right) \leq \det(A^T X B)$ by means of (2.2). Using some basic properties of the determinant we obtain (2.1).

By Theorem 2.1, the equality in (2.3) holds if and only if the matrix $A^T X B$ is symmetric, that is, $C = 2 A^T X B$. \square

Remark 2.3 Let $A, B, X \in M_n$ and let C be a $n \times n$ positive definite matrix. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ and $\delta_1, \dots, \delta_n$ be the eigenvalues of A, B, C and X , respectively.

If the matrix equation (1.1) is consistent, then the equality in (2.1) holds if and only if $C = 2 A^T X B$. For this case, $\det A \det B \neq 0$, since $\det C > 0$, and

$$\frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} = \prod_{i=1}^n \delta_i.$$

Next, we study the case when the inequality in (2.1) is strict.

Theorem 2.4 Let $A, B, X \in M_n$ and let C be a $n \times n$ positive definite matrix, such that $C \neq 2A^T X B$. If the matrix equation (1.1) is consistent, then

$$\frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} < \prod_{i=1}^n \delta_i,$$

if and only if $\prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i > 0$, where $\rho(A) = \{\alpha_1, \dots, \alpha_n\}$, $\rho(B) = \{\beta_1, \dots, \beta_n\}$, $\rho(C) = \{\gamma_1, \dots, \gamma_n\}$ and $\rho(X) = \{\delta_1, \dots, \delta_n\}$.

Proof Let C be a $n \times n$ positive definite matrix such that $C \neq 2A^T X B$. Suppose the matrix equation (1.1) is consistent. Applying Theorem 2.2, we obtain $\det(C) < 2^n \det(A) \det(X) \det(B)$.

(\Leftarrow) Suppose $\prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i > 0$. As $\det(A) = \prod_{i=1}^n \alpha_i$ and $\det(B) = \prod_{i=1}^n \beta_i$, we have

$$\frac{\det C}{2^n \det A \det B} < \det X,$$

since $\det A \det B > 0$. As the determinant of a matrix is equal to the product of its eigenvalues the result follows.

(\Rightarrow) Suppose now

$$\prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i \leq 0, \quad (2.4)$$

and

$$\frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} < \prod_{i=1}^n \delta_i. \quad (2.5)$$

By hypothesis C is a positive definite matrix, then $\det(A) \det(B) \neq 0$. Hence, the equality in (2.4) can not occur and

$$\frac{\det C}{2^n \det A \det B} > \det X, \quad (2.6)$$

since $\det A \det B < 0$. Equation (2.6) is equivalent to

$$\frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} > \prod_{i=1}^n \delta_i. \quad (2.7)$$

By (2.5) and (2.7), we obtain

$$\prod_{i=1}^n \delta_i < \frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} < \prod_{i=1}^n \delta_i,$$

a contradiction. So we have proved the theorem. \square

The next result was firstly shown by Dehghan and Hajarian in [4] and is a consequence of Theorem 2.4.

Corollary 2.5 Let $A, B, X \in M_n$ and let C be a $n \times n$ positive definite matrix. If the matrix equation (1.1) is consistent and A and B are stability matrices, then

$$\frac{\prod_{i=1}^n \gamma_i}{2^n \prod_{i=1}^n \alpha_i \prod_{i=1}^n \beta_i} \leq \prod_{i=1}^n \delta_i. \quad (2.8)$$

Proof If $C = 2A^T X B$, then the equality in (2.8) occurs by Remark 2.3. Consider, now, $C \neq 2A^T X B$. Since A and B are stability matrices, then $\det A \det B > 0$ and the result easily follows from Theorem 2.4. \square

Remark 2.6 Dehghan and Hajarian, [4], obtained similar results for matrix equation (1.2) considering X an unknown symmetric matrix. We note that, applying the same techniques as above we can obtain similar results as Theorem 2.2 and Theorem 2.4 generalizing Theorem 2.2 and Theorem 2.4 in [4].

3 Inequalities Involving the Trace

In the sequel, M_n denotes the algebra of $n \times n$ complex matrices and U^* denotes the Hermitian adjoint of $U \in M_n$, defined by $U^* = \overline{U}^T$, where \overline{U} is the component-wise conjugate of U . We say U is an unitary matrix if $U^*U = I_n$. A matrix $A \in M_n$ is normal if $A^*A = AA^*$. If $A \in M_n$ is a normal matrix with real eigenvalues, then A is a Hermitian matrix.

Given $C \in M_n$, the C -numerical range of $A \in M_n$, is as a connected and compact subset of \mathbb{C} defined by

$$W_C(A) = \{ \text{Tr}(CU^*AU) : U \text{ is unitary} \}. \quad (3.1)$$

This concept was introduced by Goldberg and Straus in [1] and is an useful concept in studying properties that are invariant under unitary similarities, since if $A = U^*A'U$ and $C = V^*C'V$, where U, V are unitary matrices, then $W_C(A) = W_{C'}(A')$.

The following well known set of points on the complex plane is important in our investigation.

Definition 3.1 Let S_n be the symmetric group of degree n . We define the σ -points of $W_C(A)$ by

$$z_\sigma = \sum_{i=1}^n \gamma_i \alpha_{\sigma(i)}, \quad \sigma \in S_n,$$

where $\alpha_1, \dots, \alpha_n$ and $\gamma_1, \dots, \gamma_n$ are the eigenvalues of A and C , respectively.

It can be easily seen that all the $n!$ σ -points (not necessarily distinct) belong to $W_C(A)$ and if A and C are normal matrices, then

$$W_C(A) = \text{Co} \{ z_\sigma : \sigma \in S_n \},$$

where $\text{Co} \{ \cdot \}$ denotes the convex hull of the set $\{ \cdot \}$.

Lemma 3.2 Let $A, B, X, C \in M_n$. If the matrix equation (1.1) is consistent, then

$$\text{Tr}(A^T X B) = \frac{1}{2} \text{Tr}(C).$$

Proof Let $Y = A^T X B - B^T X^T A$. Considering the matrix equation (1.1), we can write

$$\begin{aligned} A^T X B &= \frac{1}{2}(A^T X B + B^T X^T A) + \frac{1}{2}(A^T X B - B^T X^T A) \\ &= \frac{1}{2}C + \frac{1}{2}Y. \end{aligned}$$

Since Y is skew-symmetric matrix, then $\text{Tr}(Y) = 0$ and the result follows. \square

Next, we present the main result of this section.

Theorem 3.3 Let $C \in M_n$ and $A, B, X \in M_n$ be normal matrices, such that $A^T B = B A^T$. If the matrix equation (1.1) is consistent, then

$$\min_{\sigma} |z_{\sigma}| \leq \frac{1}{2} |\text{Tr}(C)| \leq \max_{\sigma} |z_{\sigma}|,$$

where $z_{\sigma} = \sum_{i=1}^n \alpha_i \beta_i \delta_{\sigma(i)}$, $\sigma \in S_n$, the permutation group of degree n , and $\rho(A) = \{\alpha_1, \dots, \alpha_n\}$, $\rho(B) = \{\beta_1, \dots, \beta_n\}$, $\rho(C) = \{\gamma_1, \dots, \gamma_n\}$ and $\rho(X) = \{\delta_1, \dots, \delta_n\}$.

Proof Since commuting normal matrices maybe simultaneously diagonalizable (see [2, Theorem 2.55]), then there exists a unitary matrix $U \in M_n$, such that

$$A^T = U^* \Lambda_A U \quad \text{and} \quad B = U^* \Lambda_B U,$$

where Λ_A and Λ_B are diagonal matrices with diagonal entries $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n , respectively. The trace is invariant under cyclic permutations, hence

$$\text{Tr}(A^T X B) = \text{Tr}(\Lambda_A \Lambda_B U X U^*).$$

As $\text{Tr}(A^T X B) \in W_{\Lambda_A \Lambda_B}(X)$ and X is a normal matrix, then $\text{Tr}(A^T X B)$ can be written as

$$\text{Tr}(A^T X B) = \sum_{\sigma \in S_n} u_{\sigma} z_{\sigma} \quad \text{with} \quad z_{\sigma} = \sum_{i=1}^n \alpha_i \beta_i \delta_{\sigma(i)}$$

with $0 \leq u_{\sigma} \leq 1$ and $\sum_{\sigma \in S_n} u_{\sigma} = 1$. It can be easily seen that

$$\min_{\sigma} |z_{\sigma}| \leq |\text{Tr}(A^T X B)| \leq \max_{\sigma} |z_{\sigma}|$$

and the result follows by Lemma 3.2. \square

Theorem 3.4 Let $C \in M_n$, $A, B, X \in M_n$ be Hermitian matrices, such that $A^T B = B A^T$. Let S_n be the permutation group of degree n . If the matrix equation (1.1) is consistent, then there exists $\sigma \in S_n$, such that

$$\sum_{i=1}^n \alpha_{\sigma(i)} \beta_{\sigma(i)} \delta_{n-i} \leq \frac{1}{2} \sum_{i=1}^n \gamma_i \leq \sum_{i=1}^n \alpha_{\sigma(i)} \beta_{\sigma(i)} \delta_i,$$

where $\rho(A) = \{\alpha_1, \dots, \alpha_n\}$, $\rho(B) = \{\beta_1, \dots, \beta_n\}$, $\rho(C) = \{\gamma_1, \dots, \gamma_n\}$ and $\rho(X) = \{\delta_1, \dots, \delta_n\}$, such that $\alpha_{\sigma(1)} \beta_{\sigma(1)} \geq \dots \geq \alpha_{\sigma(n)} \beta_{\sigma(n)}$ and $\delta_1 \geq \dots \geq \delta_n$.

Proof If $A \in M_n$ and $C \in M_n$ are Hermitian matrices with eigenvalues $\alpha_1 \geq \dots \geq \alpha_n$ and $\gamma_1 \geq \dots \geq \gamma_n$, then the C -numerical range is a line segment with endpoints $\sum_{i=1}^n \gamma_i \alpha_{n-i+1}$ and $\sum_{i=1}^n \gamma_i \alpha_i$ (see [3]). In an analogous way as the proof of Theorem 3.3, we have $\text{Tr}(A^T X B) \in W_{\Lambda_A \Lambda_B}(X)$. The matrix $\Lambda_A \Lambda_B$ is a diagonal matrix with diagonal entries $\alpha_1 \beta_1, \dots, \alpha_n \beta_n$.

Let $\sigma \in S_n$ and let P_σ be a permutation matrix associated with σ , such that $P_\sigma^T \Lambda_A \Lambda_B P_\sigma$ is a diagonal matrix with principal entries $\alpha_{\sigma(1)}\beta_{\sigma(1)} \geq \cdots \geq \alpha_{\sigma(n)}\beta_{\sigma(n)}$. Due to the unitary invariance of the C -numerical range, we have $W_{\Lambda_A \Lambda_B}(X) = W_{P_\sigma^T \Lambda_A \Lambda_B P_\sigma}(X)$. The result follows now easily. \square

Remark 3.5 Considering matrix equation (1.2) for an unknown symmetric matrix X , we can obtain similar results to Theorem 3.3 and Theorem 3.4 applying the same techniques as above.

4 Conclusions

Considering the matrix equations (1.1) and (1.2) we obtained inequalities involving the determinant and the product of the eigenvalues of the solutions of these matrix equations using a similar approach as the one developed by Dehghan and Hajarian. For the same matrix equations, inspired by their work, we developed inequalities involving the trace of a $n \times n$ matrix C and the sum of product of the eigenvalues of A, B and X arranged in a given way using the theory of the numerical ranges. It would be interesting to study this theory for the case when A^T and B are not simultaneously diagonalizable. We leave this as a topic for further research.

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