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On surfaces of general type with $p_g = q = 1$ having an involution

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Resumo alargado

O principal assunto desta tese é o estudo de superfícies S algébricas e projectivas de tipo geral com $p_g = q = 1$ que têm uma involução i. Estuda-se também o caso $p_g = 1$, $q = 0$ e S/i birracional a uma superfície $K3$.

Involuções surgem em muitos contextos no estudo de superfícies algébricas. Por exemplo a não birracionalidade da aplicação bicanónica de uma superfície de tipo geral implica na maioria dos casos a existência de uma involução na superfície.

A aplicação bicanónica ϕ_2 (definida pelo sistema linear $|2K_S|$) de superfícies de tipo geral S tem sido objecto de estudo por vários autores. Se a superfície S tem uma fibração de género 2, então a aplicação bicanónica de S é necessariamente composta com uma involução. Este é o chamado *caso standard* de não birracionalidade da aplicação bicanónica. Pelos resultados de Bombieri, [Bo], refinados mais tarde por Reider, [Rd], se uma superfície minimal S com $K_S^2 > 9$ tem aplicação bicanónica não birracional, então S tem uma fibração de género 2, ou seja apresenta o caso standard.

O caso não-standard é considerado em [Du], [CaCM], [CFM], [CM1], [CM2], [Xi2] e [Br], mas o caso $p_g = q = 1$ não está ainda completamente classificado. Mais geralmente, superfícies de tipo geral com $p_g = q = 1$ não estão bem compreendidas, e poucos exemplos são conhecidos.

Se a superfície minimal S satisfaz $p_g = q = 1$, então $2 \leq K_S^2 \leq 9$ e a aplicação de Albanese é uma fibração conexa sobre uma curva elíptica. Seja g o género de uma fibra genérica da fibração de Albanese de S. Superfícies de tipo geral minimais com $p_g = q = 1$ e $K^2 = 2, 3$ são classificadas em [Ca1], [CC1] e [CC2]. Superfícies com $K^2 = 8$ que são uma fibração isotrivial standard são classificadas em [Po1] e [Po4]. Para outros valores de K^2 temos apenas os exemplos de Catanese ([Ca2]), com $(K^2, g) = (4, 2), (5, 2)$, o exemplo de Xiao ([Xi1]), com $(K^2, g) = (4, 2)$, e o exemplo de Ishida ([Is]), com $(K^2, g) = (4, 3)$.

Como já foi dito, o principal assunto desta tese é o estudo de superfícies S algébricas e projectivas de tipo geral com $p_g = q = 1$ que têm uma involução i.

Os principais resultados obtidos são:

- Teorema 3.1.2: para $q(S) = 0$, S/i birracional a uma superfície $K3$ e ϕ_2 composta com i, dá-se uma descrição de S/i como plano duplo;

- Teorema 3.2.1: classifica-se o caso $p_g(S) = q(S) = 1$, $\mathrm{Kod}(S/i) \geq 0$ e ϕ_2 composta com i. Em particular, mostra-se que a fibração de Albanese de S tem género 2 ou S/i é birracional a uma superfície $K3$;

- Proposições 5.1.1, 5.1.2 e 5.1.3: dá-se uma lista de possibilidades para o caso $p_g(S) = q(S) = 1$, $\mathrm{Kod}(S/i) \geq 0$ e ϕ_2 não composta com i;

- a obtenção de exemplos novos, listados abaixo.

Todorov, [To2], foi o primeiro a dar exemplos de superfícies S de tipo geral com $p_g = 1$, $q = 0$ e aplicação bicanónica composta com uma involução i de S tal que S/i é birracional a uma superfície $K3$, às quais chamamos *superfícies de Todorov.* Morrison, [Mo], descreve o espaço de moduli das superfícies de Todorov. Para $K^2 = 1$, Todorov mostra, em [To1], que a resolução W de S/i é um plano duplo com um modelo plano ramificado sobre duas cúbicas. Na Secção 3.1 demonstra-se que isto é verdade também para $K^2 > 1$ e dá-se um exemplo diferente dos exemplos de Todorov.

Existem dois métodos usuais de construção de superfícies: Campedelli — coberturas duplas ramificadas sobre curvas (possivelmente singulares) — e Godeaux — quocientes por acção de um grupo (*cf.* [Re1]). Nesta tese o primeiro método é utilizado na obtenção de exemplos novos. Muitas vezes as singularidades impõem condições a mais sobre os parâmetros do sistema linear de curvas, o que implica cálculos complicados. Estes cálculos são efectuados utilizando o *Sistema Algébrico Computacional MAGMA* (V2.11-14) (ver http://magma.maths.usyd.edu.au/magma para mais informação sobre o Magma).

Neste trabalho são dadas várias construções de superfícies S de tipo geral, não singulares e minimais, com $p_g = q = 1$. Em particular são obtidos os seguintes exemplos:

- $K^2 = 7$, $g = 5$, $\deg(\phi_2) \geq 2$, S um plano duplo, e
  $K^2 = 7$, $g = 3$, $\deg(\phi_2) = 1$, S uma cobertura bidupla de $\mathbb{P}^2$:
  estes são os primeiros exemplos com $K^2 = 7$ (ver Secções 4.2.2 e 6.8);

- $K^2 = 6$, $g = 3$, $\deg(\phi_2) = 2$, $\phi_2(S)$ birracional a uma superfície $K3$:
  este é o primeiro exemplo com $K^2 = 6$ (ver [Ri]) (ver Secção 3.2.2);

- $K^2 = 5$, $g = 4$, $\deg(\phi_2) \geq 2$, S um plano duplo:
  este é o primeiro exemplo com $K^2 = 5$ e $g \neq 2$ (ver Secção 4.2.2);

- $K^2 = 8$, $g = 3$: é o primeiro exemplo com $K^2 = 8$ que não é uma fibração isotrivial standard (ver Secção 6.7);

- $K^2 = 6$, $g = 4$ ou 3, $\deg(\phi_2) \geq 2$, S um plano duplo, e
  $K^2 = 6$, $g = 3$, $\deg(\phi_2) = 1$, S uma cobertura bidupla de $\mathbb{P}^2$ (ver Secções 4.2.1, 4.2.3 e 6.9);

- $K^2 = 4$, $g = 3$, $\deg(\phi_2) \geq 2$, S um plano duplo (ver Secção 4.2.1);

- $K^2 = 4$, $g = 2$, $\deg(\phi_2) = 2$, S uma cobertura bidupla de $\mathbb{P}^2$
  (temos $\deg(\phi_2) = 4$ no exemplo de Catanese em [Ca2]) (ver Secção 6.6);

- $K^2 = 3$, $g = 3$, $\deg(\phi_2) \geq 2$, S um plano duplo (ver Secção 4.2.2).

Em [Po4], Polizzi classifica superfícies de tipo geral com $p_g = q = 1$, $K^2 = 8$ e aplicação bicanónica de grau 2. Dá exemplos usando quocientes por acção de um grupo e mostra que estas superfícies são planos duplos de Du Val, descrevendo a curva de ramificação de um modelo plano correspondente (ver Teorema 1.3.2). Nas Secções 4.2.1 e 4.2.4 mostra-se como obter equações para tais curvas de ramificação. Na Secção 6.12 descreve-se como obter uma equação de um plano duplo com $K^2 = 8$ cuja aplicação bicanónica não é composta com a involução associada.

Na secção 3.2.2, é construída uma superfície minimal de tipo geral com $p_g = q = 1$, $K^2 = 6$ e $g = 3$ tal que $\deg(\phi_2) = 2$ e $\phi_2(S)$ é uma superfície $K3$. Este exemplo contradiz um resultado de Xiao Gang. Mais precisamente, a lista de possibilidades em [Xi2] exclui o caso em que S não tem uma fibração de género 2, $p_g(S) = q(S) = 1$ e S/i é birracional a uma superfície $K3$. No Lema 7 de [Xi2] está escrito que R tem apenas singularidades negligíveis, mas aqui falha a possibilidade $\chi(K_{\widetilde{P}} + \widetilde{\delta}) < 0$ na fórmula (3) da página 727. De facto iremos ver que R ($\overline{B}$ na nossa notação) pode ter uma singularidade não negligível.

Esta tese está organizada como segue. No Capítulo 2 demonstram-se as Proposições 2.1.2, 2.1.3 e 2.1.4. Para uma superfície S com uma involução i, estas proposições relacionam os invariantes de S e S/i com a curva de ramificação da cobertura $S \to S/i$, as suas singularidades e o número de nodos de S/i. Aqui são importantes as bem conhecidas fórmulas para coberturas duplas; é utilizado um resultado de Miyaoka sobre o número máximo de curvas racionais disjuntas numa superfície. Também se descreve a acção de i na fibração de Albanese.

No Capítulo 3 estuda-se o caso ϕ_2 composta com i e $\mathrm{Kod}(S/i) \geq 0$. A Secção 3.1 contém uma descrição de S com $q(S) = 0$. O Teorema 1.1 (*cf.* [St]) é importante nesta secção. Para $q(S) \neq 0$, prova-se um teorema de classificação na Secção 3.2. A existência da fibração de Albanese e as equações da Proposição 2.1.2 são ingredientes cruciais na demonstração.

Tal como notamos acima, se ϕ_2 é composta com i e $\mathrm{Kod}(S/i) = -\infty$, então S é um plano duplo de Du Val (*cf.* Teorema 1.2.4). No Capítulo 4 obtêm-se exemplos de tais superfícies. Mais precisamente, explica-se como obter equações de planos duplos de tipo geral com $p_g = q = 1$, $K^2 = 2, \ldots, 8$, $g \neq 2$ (se $K^2 \neq 2$) e aplicação bicanónica não birracional. As construções com $K^2 = 5, 6, 7$ produzem exemplos não referidos na literatura, enquanto as outras dão uma descrição alternativa de superfícies possivelmente já conhecidas. Para obter equações de curvas de ramificação são impostas condições sobre os parâmetros lineares de sistemas lineares de curvas planas. Os cálculos são efectuados usando o sistema computacional Magma, no Apêndice A. Este apêndice contém também outros cálculos utilizando o Magma, relacionados com o Capítulo 6 e a Secção 3.2.2.

O Capítulo 5 descreve as possibilidades para o caso ϕ_2 não composta com i. Para este estudo são importantes as Proposições 2.1.2, 2.1.3 e 2.1.4. Vários exemplos são obtidos no Capítulo 6, como coberturas biduplas de superfícies (com uma excepção, um plano duplo).

Abstract

The main subject of this thesis is the study of surfaces of general type S with $p_g = q = 1$ having an involution i. For such surfaces one has $2 \leq K_S^2 \leq 9$ and only few examples with $K^2 = 2, \ldots, 5$ or 8 are known.

The quotient surface S/i is a surface with $p_g \leq 1$ and $q \leq 1$ and its Kodaira dimension, $\mathrm{Kod}(S/i)$, can be any.

A list of possibilities for the case $\mathrm{Kod}(S/i) = -\infty$ and bicanonical map ϕ_2 composed with i has been given by Xiao in [Xi2]. Here the computational algebra system Magma is used to compute equations of plane models of double planes with $p_g = q = 1$ and $K^2 = 2, \ldots, 8$.

For $\mathrm{Kod}(S/i) \geq 0$ and ϕ_2 composed with i, we show that S/i is regular and either: a) the Albanese fibration of S is of genus 2 or b) S has no genus 2 fibration and S/i is birational to a $K3$ surface. For case a) a list of possibilities and examples are given. An example for case b) with $K^2 = 6$ is constructed. This last case was a possibility mistakenly excluded in [Xi2].

For the case ϕ_2 not composed with i, a list of possibilities is given and several new examples are obtained, mostly as bidouble covers of surfaces. In particular minimal surfaces of general type with $p_g = q = 1$, $K^2 = 6, 7$ and birational bicanonical map are constructed.

The case $p_g = 1$, $q = 0$ and S/i birational to a $K3$ surface is also considered. It is shown that the smooth minimal model W of S/i is a double plane, with a plane model ramified over two cubics.

Contents

Introduction

The main subject of this thesis is the study of surfaces of general type S with $p_g = q = 1$ having an involution i. We also study surfaces with $p_g = 1$ and $q = 0$ having an involution i such that the quotient surface S/i is birational to a $K3$ surface.

Involutions appear in many contexts in the study of algebraic surfaces. For instance in most cases the non birationality of the bicanonical map for surfaces implies the existence of an involution on the surface.

The bicanonical map ϕ_2 of a surface S (given by $|2K_S|$) has been considered by several authors. There is an instance where the bicanonical map is necessarily composed with an involution i of S (*i.e.* where ϕ_2 factors through $S \to S/i$): if f is a morphism from S to a curve such that a general fibre F of f is irreducible of genus 2, then the system $|2K_S|$ cuts out on F a subseries of the bicanonical series of F, which is composed with the hyperelliptic involution of F, and then ϕ_2 is composed with an involution. This is the so called *standard case* of non-birationality of the bicanonical map.

By the results of Bombieri, [Bo], improved later by Reider, [Rd], a minimal surface S satisfying $K^2 > 9$ and ϕ_2 non-birational necessarily presents the standard case of non-birationality of the bicanonical map. Several authors have studied the non-standard case.

Du Val, [Du], classified the regular surfaces S of general type with $p_g \geq 3$ whose general canonical curve is smooth and hyperelliptic. Of course, for these surfaces, the bicanonical map is composed with an involution i such that S/i is rational. The families of surfaces exhibited by Du Val, presenting the non-standard case, are nowadays called the *Du Val examples*.

Other authors have later studied the non-standard case: the articles [CaCM], [CFM], [CM1], [CM2], [Xi2] and [Br] treat the cases $\chi(\mathcal{O}_S) > 1$ or $q(S) \geq 2$ (*cf.* the expository paper [Ci] for more information on this problem).

Xiao Gang, [Xi2], presented a list of possibilities for the non-standard case of non-birationality of the bicanonical morphism ϕ_2. For the case when ϕ_2 has degree 2 and the bicanonical image is a ruled surface, Theorem 2 of [Xi2] extended Du Val's list to $p_g(S) \geq 1$ and added two extra families (this result is still valid assuming only that ϕ_2 is composed with an involution such that the quotient surface is a ruled surface). Later G. Borrelli, [Br], excluded these two families, confirming that the only possibilities for this case are the Du Val examples.

For the case when the bicanonical image is a non-ruled surface, results in the papers [Xi2], [CaCM] and [CM2] give a list of possibilities (see Theorem 1.2.1), but the case $p_g = q = 1$ is not completely described. More in general, surfaces of general type with $p_g = q = 1$ are still not well understood, and few examples are known.

For a minimal surface S satisfying $p_g(S) = q(S) = 1$, one has $2 \leq K_S^2 \leq 9$ and the Albanese map is a connected fibration onto an elliptic curve. We denote by g the genus of a general Albanese fibre of S.

A classification for the surfaces with $K^2 = 2, 3$ has been obtained by Catanese and Ciliberto ([Ca1], [CC1], [CC2]). Surfaces with $K^2 = 8$ that are isogenous to a product of curves have been studied by Polizzi ([Po4], [Po1]), who also looked to the case $K^2 = 3$ ([Po3]). For other values of K^2 one has the examples of Catanese ([Ca2]), with $(K^2, g) = (4, 2), (5, 2)$, and Xiao ([Xi1]), with $(K^2, g) = (4, 2)$ (thus having a genus 2 pencil), and the Ishida example ([Is]) with $(K^2, g) = (4, 3)$.

As it was said in the beginning, the main subject of this thesis is the study of surfaces of general type S with $p_g = q = 1$ having an involution i.

The main results obtained are:

- Theorem 3.1.2: for $q(S) = 0$, S/i birational to a $K3$ surface and ϕ_2 composed with i, a double plane description of S/i is given;

- Theorem 3.2.1: the case $p_g(S) = q(S) = 1$, $\mathrm{Kod}(S/i) \geq 0$ and ϕ_2 composed with i is classified. In particular it is shown that the Albanese fibration of S is of genus 2 or S/i is birational to a $K3$ surface;

- Propositions 5.1.1, 5.1.2 and 5.1.3: a list of possibilities for the case $p_g(S) = q(S) = 1$, $\mathrm{Kod}(S/i) \geq 0$ and ϕ_2 not composed with i is given;

- the construction of new examples, listed below.

Todorov, [To2], was the first to give examples of surfaces S of general type with $p_g = 1$ and $q = 0$ having bicanonical map composed with an involution i of S such that S/i is birational to a $K3$ surface, which we call *Todorov surfaces*. Morrison, [Mo], gives an explicit description of the moduli spaces of Todorov surfaces. For $K^2 = 1$, Todorov has shown, in [To1], that the smooth minimal model W of S/i is a double plane, with a plane model ramified over two cubics. In Section 3.1 it is shown that this is also true for $K^2 > 1$ and an example different from the Todorov's examples is given.

There are two typical methods to construct examples of surfaces: Campedelli — double covers ramified over curves with (complicated) singularities — and Godeaux — quotients by group actions (*cf.* [Re1]). In this thesis the first method is used to obtain new examples. Often the singularities impose too many conditions on the linear parameters, and this implies hard calculations. The *Computational Algebra System MAGMA* (V2.11-14) is used to perform the heavy computations (visit http://magma.maths.usyd.edu.au/magma/ for information about Magma).

In this work several constructions of smooth minimal surfaces S of general type with $p_g = q = 1$ are given. In particular the following examples are obtained:

- $K^2 = 7$, $g = 5$, $\deg(\phi_2) \geq 2$, S is a double plane, and
  $K^2 = 7$, $g = 3$, $\deg(\phi_2) = 1$, S is a bidouble cover of $\mathbb{P}^2$:
  these are the first examples with $K^2 = 7$ (see Sections 4.2.2 and 6.8);

- $K^2 = 6$, $g = 3$, $\deg(\phi_2) = 2$, $\phi_2(S)$ is birational to a $K3$ surface:
  this is the first example with $K^2 = 6$ (see [Ri]) (see Section 3.2.2);

- $K^2 = 5$, $g = 4$, $\deg(\phi_2) \geq 2$, S is a double plane:
  this is the first example with $K^2 = 5$ and $g \neq 2$ (see Section 4.2.2);

- $K^2 = 8$, $g = 3$: it is the first example with $K^2 = 8$ which is not a standard isotrivial fibration (see Section 6.7);

- $K^2 = 6$, $g = 4$ or 3, $\deg(\phi_2) \geq 2$, S is a double plane, and
  $K^2 = 6$, $g = 3$, $\deg(\phi_2) = 1$, S is a bidouble cover of $\mathbb{P}^2$ (see Sections 4.2.1, 4.2.3 and 6.9);

- $K^2 = 4$, $g = 3$, $\deg(\phi_2) \geq 2$, S is a double plane (see Section 4.2.1);

- $K^2 = 4$, $g = 2$, $\deg(\phi_2) = 2$, S is a bidouble cover of $\mathbb{P}^2$
  (one has $\deg(\phi_2) = 4$ for the Catanese example in [Ca2]) (see Section 6.6);

- $K^2 = 3$, $g = 3$, $\deg(\phi_2) \geq 2$, S is a double plane (see Section 4.2.2).

The recent preprint [Po2] also gives examples with $K^2 = 6, 4$ and $g \neq 2$. These surfaces contain $8 - K^2 \neq 0$ (-2)-curves, all in the same Albanese fibre. In each of the double cover constructions referred above, such an Albanese fibre must be the pullback of one of four fibres F_A^i of S/i (see Section 2.2). One can verify that, in the examples with $K_S^2 = 4$ or 6, none of the fibres F_A^i induces a configuration like the one above. Therefore these surfaces are different from Polizzi's examples.

In [Po4], Polizzi classifies surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2. He gives examples using quotients under the action of a group and shows that these surfaces are Du Val double planes, describing the branch locus of the respective plane models (see Theorem 1.3.2). Sections 4.2.1 and 4.2.4 show how to obtain equations for such branch locus. Section 6.12 describes how to obtain an equation of a double plane with $K^2 = 8$ having bicanonical map not composed with the associated involution.

A minimal smooth surface of general type with $p_g = q = 1$, $K^2 = 6$ and $g = 3$ such that ϕ_2 is of degree 2 onto a $K3$ surface is constructed in Section 3.2.2. This example contradicts a statement of Xiao Gang. More precisely, the list of possibilities in [Xi2] rules out the case where S has no genus 2 fibration, $p_g(S) = q(S) = 1$ and S/i is birational to a $K3$ surface. In Lemma 7 of [Xi2] it is written that R has only negligible singularities, but the possibility $\chi(K_{\widetilde{P}} + \widetilde{\delta}) < 0$ in formula (3) of page 727 was overlooked. In fact we will see that R ($\overline{B}$ in our notation) can have a non-negligible singularity.

This thesis is organized as follows.

In Chapter 2 we prove Propositions 2.1.2, 2.1.3 and 2.1.4. For a surface S with an involution i, these relate the invariants of S and S/i with the branch locus of the cover $S \to S/i$, its singularities and the number of nodes of S/i. The well known double cover formulas are important here; a Miyaoka's result on the maximal number of disjoint smooth rational curves on a minimal surface with Kodaira dimension ≥ 0 is used. We also describe the action of the involution i on the Albanese fibration of S.

In Chapter 3 we study the case ϕ_2 composed with i and $\mathrm{Kod}(S/i) \geq 0$. Section 3.1 contains a description for S with $q(S) = 0$. Theorem 1.1 ([St]) is important in the proof. For $q(S) \neq 0$, a classification theorem is proved in Section 3.2. Crucial ingredients are the existence of the Albanese fibration and the formulas of Proposition 2.1.2.

As noted above, if ϕ_2 is composed with i and $\mathrm{Kod}(S/i) = -\infty$, then S is a Du Val double plane (*cf.* Theorem 1.2.4). In Chapter 4 we construct explicit examples of such surfaces, more precisely we explain how to obtain equations of double planes of general type with $p_g = q = 1$, $K^2 = 2, \ldots, 8$, $g \neq 2$ (if $K^2 \neq 2$) and non-birational bicanonical map. The constructions with $K^2 = 5, 6, 7$ yield previously unknown examples of surfaces, whilst the others give an alternative description of (possibly) known surfaces. To obtain the equations of the branch loci we impose conditions to the linear parameters of systems of plane curves. Since the ramification curves are contained in Albanese fibres, in some cases it is easier to start by constructing non-reduced Albanese fibres, which simplify the computations. The calculations are done using the computational system Magma in Appendix A. This appendix also contains other Magma computations, related to Chapter 6 and Section 3.2.2.

In Chapter 5, a list of possibilities for the case ϕ_2 not composed with i is given. The proof consists mainly in examining the possibilities allowed by Propositions 2.1.2, 2.1.3 and 2.1.4. Several examples are obtained in Chapter 6, mostly as bidouble covers of surfaces.

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface S, the *canonical class* is denoted by K_S, the *geometric genus* by $p_g(S) := h^0(S, \mathcal{O}_S(K_S))$, the *irregularity* by $q(S) := h^1(S, \mathcal{O}_S(K_S))$, the *Euler characteristic* by $\chi(\mathcal{O}_S) = 1 + p_g(S) - q(S)$ and the *Kodaira dimension* of S is denoted by $\mathrm{Kod}(S)$.

We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by $\equiv$. A $(-n)$-*curve* C on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -n$. A *node* is an ordinary singularity of order 2 and a *nodal curve* is a (-2)-curve. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. A $(m_1, m_2, \ldots)$- -point, or point of order $(m_1, m_2, \ldots)$, is a point of multiplicity m_1, which resolves to a point of multiplicity m_2 after one blow-up, etc.

The surface Y is a *double cover* of the surface X if there is a finite degree 2 morphism $Y \to X$. An *involution* of a surface S is an automorphism of S of order 2. We say that a map is *composed with an involution* i of S if it factors through the double cover $S \to S/i$.

The rest of the notation is standard in algebraic geometry (following the books [Be], [BPV], [GH] or [Ha]).

Chapter 1

Preliminaries

1.1 General facts

Canonical resolution

This is an important technical tool that is used several times through the text. Given a double cover $\pi : Y \to X$, with X a smooth surface, this method reduces the problem of resolving the singularities of Y to the resolution of the singularities of the branch locus of π. See [BPV] for more information.

Bicanonical map

The *bicanonical map* of a surface of general type S,

$$\phi_2 : S \dashrightarrow \mathbb{P}^{\dim|2K_S|},$$

is the map defined by the linear system $|2K_S|$, where K_S denotes a canonical divisor of S. In 1985 Xiao Gang proved the following

Theorem 1.1.1 ([Xi1]) *Let S be a minimal surface of general type with $p_g \geq 1$, or $p_g = 0$ and $K^2 \geq 2$. Then $|2K_S|$ is not composed with a pencil, i.e. the image of ϕ_2 is a surface.*

The next theorem is a consequence of results of Francia, Reider, Catanese and Ciliberto. For more details see [Ci].

Theorem 1.1.2 ([Ca1], [Fr], [Rd], [CC1]) *Let S be a minimal surface of general type with $p_g \geq 1$. Then ϕ_2 is a morphism, i.e. $|2K_S|$ is base point free.*

Todorov surfaces

In [To2] Todorov gives a construction of minimal surfaces of general type S with $p_g = 1$, $q = 0$, $K^2 = 8 - j$, $j \in \{0, \dots, 6\}$, and bicanonical map composed with an involution i such that S/i is birational to a $K3$ surface. We give a brief description of this.

Consider a Kummer surface Q in $\mathbb{P}^3$, *i.e.* a quartic having as only singularities 16 nodes a_i. Choose $a_1, \dots, a_6$ in general position and let G be the intersection of Q with a general quadric through j of the nodes $a_1, \dots, a_6$. Let $\widetilde{Q}$ be the minimal resolution of Q and $\widetilde{G} \subset \widetilde{Q}$ be the strict transform of G. The surface S is the minimal model of the double cover of $\widetilde{Q}$ ramified over $\widetilde{G} + \sum_{j+1}^{16} A_i$, where $A_i \subset \widetilde{Q}$ are the (-2)-curves which contract to the nodes a_i.

Hyperelliptic linear systems on a $K3$

The next result follows from [St, (4.1), Theorem 5.2, Propositions 5.6 and 5.7].

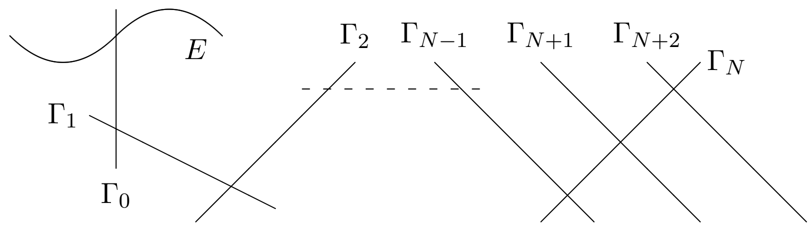
Theorem 1.1.3 ([St]) *Let $|D|$ be a complete linear system on a smooth $K3$ surface F, without fixed components and such that $D^2 \geq 4$. Denote by φ_D the map given by $|D|$. If φ_D is non-birational and the surface $\varphi_D(F)$ is singular then there exists an elliptic pencil $|E|$ such that $ED = 2$ and one of these cases occur:*

(i) $D = O_F(4E + 2\Gamma)$ *where Γ is a smooth rational irreducible curve such that* $\Gamma E = 1$. *In this case $\varphi_D(F)$ is a cone over a rational normal twisted quartic in $\mathbb{P}^4$;*

(ii) $D = O_F(3E + 2\Gamma_0 + \Gamma_1)$, *where Γ_0 and Γ_1 are smooth rational irreducible curves such that* $\Gamma_0 E = 1$, $\Gamma_1 E = 0$ *and* $\Gamma_0\Gamma_1 = 1$. *In this case $\varphi_D(F)$ is a cone over a rational normal twisted cubic in $\mathbb{P}^3$;*

(iii) **a)** $D = O_F(2E + \Gamma_0 + \Gamma_1)$, *where Γ_0 and Γ_1 are smooth rational irreducible curves such that* $\Gamma_0 E = \Gamma_1 E = 1$ *and* $\Gamma_0\Gamma_1 = 0$;

b) $D = O_F(2E + \Delta)$, *with* $\Delta = 2\Gamma_0 + \dots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ *($N \geq 0$), where the curves Γ_i are irreducible rational curves as follows:*



In both cases $\varphi_D(F)$ is a quadric cone in $\mathbb{P}^3$.

Moreover in all the cases above the pencil $|E|$ corresponds under the map φ_D to the system of generatrices of $\varphi_D(F)$.

Invertible sheafs on m-connected curves

A curve D is said to be *m-connected* if every decomposition $D = D_1 + D_2$ satisfies $D_1 D_2 \geq m$.

Proposition 1.1.4 ([CFM, Proposition (A.5), (ii)]) *Let D be a reducible m-connected curve ($m \geq 1$) on a surface S and let $\mathcal{L}$ be an invertible sheaf on D such that* $\deg \mathcal{L}_{|\Gamma} \geq 0$, *for every component Γ of D. Let $n :=$* $\deg \mathcal{L}$.

If $n < m$ and if there is some component Δ of D such that $\deg \mathcal{L}_{|\Delta} = 0$, *then* $h^0(D, \mathcal{L}) \leq 1$.

Counting rational and elliptic curves on surfaces

Proposition 1.1.5 ([Mi, Proposition 2.1.1]) *Let X be a minimal surface of non-negative Kodaira dimension. Then the number of disjoint smooth rational curves in X is bounded by*

$$8\left(\chi(\mathcal{O}_X) - \frac{1}{9}K_X^2\right).$$

Proposition 1.1.6 ([Sa]) *Let S be a minimal smooth surface of general type and $C \subset S$ be a disjoint union of smooth elliptic curves. Then*

$$-C^2 \leq 36\chi(\mathcal{O}_S) - 4K_S^2.$$

Proof: This follows from the inequality of [Sa, Corollary 7.8], using
$KC + C^2 = 2p_a(C) - 2 = 0$. $\square$

Du Val double planes

We say that a smooth surface S is a *double plane* if S has an involution i such that S/i is a rational surface. A *plane model* of S is a double cover $X \to \mathbb{P}^2$ such that X is a normal surface and there exists a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ S/i & \longrightarrow & \mathbb{P}^2 \end{array}$$

such that the horizontal arrows denote birational maps.

Let C_0 and F denote, respectively, the negative section and a fibre of the Hirzebruch surface $\mathbb{F}_2$.

Definition 1.1.7 *The Bombieri-Du Val surface is the minimal model of a double cover of* $\mathbb{F}_2$ *with branch locus a smooth curve in* $C_0 + |7C_0 + 14F|$.

Definition 1.1.8 *Let p denote a point in $\mathbb{P}^2$ and $T_1, \ldots, T_n$ denote distinct lines through p. A Du Val surface is either*

$\mathcal{B}$) the Bombieri-Du Val surface

or a minimal double plane having a plane model with branch locus D one of the following:

$\mathcal{D}$) a smooth curve of degree 8;

$\mathcal{D}_n$) $D = D' + T_1 + \cdots + T_n$, $n \in \{0, \ldots, 6\}$, where D' is a curve of degree $10 + n$ whose non-negligible singularities are: a point of multiplicity $n + 2$ at p and a $(4, 4)$-point in T_i, with tangent T_i, $i = 1, \ldots, n$;

or one of $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{D}_n$, imposing additional 4-uple or $(3, 3)$-points to D.

Surfaces of type $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{D}_n$ are called *Du Val's ancestors.*

The Bombieri surface has $K^2 = 9$, $p_g = 6$ and $q = 0$; a Du Val ancestor of type $\mathcal{D}$ has $K^2 = 2$, $p_g = 3$ and $q = 0$; those of type $\mathcal{D}_n$ have $K^2 = 8$, $p_g = 6 - n$ and $q = 0$, except possibly in the case $n = 6$, where $p_g = q = 1$ if the 6 singular points are contained in a conic.

Notice that the imposition of a 4-uple point to the branch locus decreases K^2 by 2 and the Euler characteristic χ by 1, while a $(3, 3)$-point decreases both K^2 and χ by 1. Negligible singularities in the branch locus do not change these invariants.

The imposition of $6-n$ 4-uple or $(3,3)$-points to the branch locus of a Du Val surface of type $\mathcal{D}_n$ gives $p_g=0$ if the 6 singular points, different from p, are not contained in a conic and $p_g=q=1$ otherwise.

The bicanonical map of a Du Val surface factors through a map of degree 2 onto a rational surface. For more information on Du Val surfaces see [Du], [Ci] and [Br].

Du Val surfaces are also called *Du Val double planes*.

Bidouble covers

A bidouble cover is a finite flat Galois morphism with Galois group $\mathbb{Z}_2^2$.

To define a $\mathbb{Z}_2^2$ cover $\psi : V \to X$, with V, X smooth surfaces, one must present:

· smooth divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;

· line bundles L_1, L_2, L_3 such that $2L_g \equiv D_j + D_k$ for each permutation (g,j,k) of $(1,2,3)$.

If $\mathrm{Pic}(X)$ has no 2-torsion then the L_i's are uniquely determined by the D_i's.

Let $N := 2K_X + \sum_1^3 L_i$. One has (*cf.* [Ca2] or [Pa]):

$$p_g(V) = p_g(X) + \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)),$$

$$\chi(\mathcal{O}_V) = 4\chi(\mathcal{O}_X) + \frac{1}{2}\sum_1^3 L_i(K_X + L_i),$$

$$2K_V \equiv \psi^*(N)$$

and

$$H^0(V, \mathcal{O}_V(2K_V)) \simeq H^0(X, \mathcal{O}_X(N)) \oplus \bigoplus_{i=1}^3 H^0(X, \mathcal{O}_X(N - L_i)).$$

The bicanonical map of V is composed with the involution i_g, associated with L_g, if and only if

$$h^0(X, \mathcal{O}_X(2K_X + L_g + L_j)) = h^0(X, \mathcal{O}_X(2K_X + L_g + L_k)) = 0.$$

For more information on bidouble covers see [Ca2] or [Pa].

1.2 Classification

From [Xi2, Theorem 3], [CaCM, Theorems A and B] and [CM2, Theorem 1.1], one has the following:

Theorem 1.2.1 ([Xi2], [CaCM], [CM2]) *Let S be a minimal smooth surface of general type with $p_g \geq 1$ having an involution i such that the bicanonical map ϕ_2 of S is composed with i.*

If $\mathrm{Kod}(S/i) \geq 0$ *then either S has a genus 2 fibration or we are in one of the following cases:*

a) $p_g = 1, q = 0$, $K^2 \leq 8$, *S/i is birational to a $K3$ surface;*

b) $p_g = q = 1$, $3 \leq K^2 \leq 8$, *S/i is birational to a $K3$ surface;*

c) $p_g = q = 1$, $K^2 = 3$ *or* 4, *S/i is birational to an Enriques surface;*

d) $p_g = q = 1$, $3 \leq K^2 \leq 6$, $p_g(S/i) = 1$, $q(S/i) = 0$, $\mathrm{Kod}(S/i) = 1$;

e) $p_g = q = 2$, $K^2 = 4$;

f) $p_g = q = 3$, $K^2 = 6$.

Remark 1.2.2 *Examples for* a) *are given in* [To2]*. Cases* e) *and* f) *are completely classified in* [CM2] *and* [CaCM]*, respectively. In this thesis we exclude cases* c) *and* d) *and we give an example for* b) *with $K^2 = 6$. The existence of examples for* b) *with $K^2 \neq 6$ is an open problem.*

Remark 1.2.3 *Theorem 3 of [Xi2] contains the assumption* $\deg(\phi_2) = 2$*, but the result is still valid assuming only that ϕ_2 is composed with an involution.*

The proof of the following result was essentially done by Xiao Gang and completed by G. Borrelli.

Theorem 1.2.4 ([Xi2], [Br]) *Let S be a minimal smooth surface of general type, without a genus 2 fibration, with an involution i such that* $\mathrm{Kod}(S/i) = -\infty$.

The bicanonical map of S is composed with i if and only if S is a Du Val surface (in particular S/i is rational).

Remark 1.2.5 *If one allows S to have a genus 2 fibration, then S/i can be non-rational: see* [Ca2, Example 8] *or* Section 6.6*, where a minimal surface with $p_g = q = 1$ and Albanese fibration of genus 2 is constructed as a double cover of an irregular ruled surface.*

1.3 Surfaces with $p_g = q = 1$

Let S be a minimal smooth projective surface of general type satisfying $p_g(S) = q(S) = 1$. Note that then $2 \leq K_S^2 \leq 9$: we have $K_S^2 \leq 9\chi(\mathcal{O}_S)$, by the Myiaoka-Yau inequality (see [BPV, Chapter VII, Theorem (4.1)]), and $K_S^2 \geq 2p_g$, because S is an irregular surface (see [De]).

Furthermore, if the bicanonical map of S is not birational, then $K_S^2 \neq 9$. In fact otherwise S would have a genus 2 fibration, by [CM2], but Théorème 2.2 of [Xi1] implies that if S has a genus 2 fibration and $p_g(S) = q(S) = 1$ then $K_S^2 \leq 6$.

The cases $K_S^2 = 2$ or 3 were classified by Catanese and Ciliberto in [Ca1], [CC1] and [CC2]. In particular they proved:

Theorem 1.3.1 ([Ca1], [CC1], [CC2]) *Let S be a surface of general type with $p_g = q = 1$ and $K_S^2 = 2$ or 3. Denote by $E(n)$ the n-th symmetric product of an elliptic curve E and let g be the genus of a general Albanese fibre of S.*

Then the relative canonical map of S coincides with the paracanonical map $\gamma : S \to E(g)$ and:

a) *If $K_S^2 = 2$, then $g = 2$ and γ is a $2:1$ morphism;*

b) *If $K_S^2 = 3$ and $g = 2$, then γ is a $2:1$ rational map with exactly one base point;*

c) *If $K_S^2 = 3$ and $g \neq 2$, then $g = 3$ and γ is a morphism which is birational onto its image and it is an isomorphism of the canonical model of S onto $\gamma(S)$.*

Polizzi worked on the cases $K_S^2 = 3$, in [Po3], and $K_S^2 = 8$, in [Po4]. For $K_S^2 = 8$ and bicanonical map of degree 2, he obtained the following classification:

Theorem 1.3.2 [Po4] *Let S be a minimal surface of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2. Then S is a double plane and there exists a plane model of S with branch curve B such that:*

1. *$B = C_{16} + T_1 + \ldots + T_6$, where C_{16} is a curve of degree 16 and $T_1, \ldots, T_6$ are distinct lines passing through a point P;*

2. *The singularities of C_{16} are:*

- *a point of multiplicity 8 at P;*
- *six points of type $(4,4)$, each one tangent to T_i at a point R_i;*

3. *The curve C_{16} looks as follows:*

i) $C_{16} = C_8^1 + C_8^2$, *where each* C_8^j *is an irreducible curve of degree* 8 *having a* 4*-uple point at* P *and a* $(2,2)$*-point (ordinary tacnode) tangent to* T_i *at* R_i*, for any* i*;*

ii) *or* $C_{16} = C_4 + C_{12}$*, where each* C_j *is an irreducible curve of degree* j *with a* $j/2$*-uple point at* P *and a* $(j/4, j/4)$*-point tangent to* T_i *at* R_i*, for any* i*;*

iii) *or* C_{16} *is irreducible with a* 8*-uple point at* P *and a* $(4,4)$*-point tangent to* T_i *at* R_i*, for any* i*;*

4. *There is exactly one conic containing the points* $R_1, \ldots, R_6$.

For the remaining cases, with $K_S^2 = 4, \ldots, 7$ or 9, one has the examples of Catanese ([Ca2]), with $(K^2, g) = (4,2), (5,2)$, the example of Xiao ([Xi1]), with $(K^2, g) = (4,2)$, and the Ishida example ([Is]) with $(K^2, g) = (4,3)$.

Chapter 2

Results on involutions

2.1 General facts

Let S be a smooth minimal surface of general type with an involution i. Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let S/i be the quotient of S by i and $p : S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of S at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution i induces a biregular involution $\widetilde{i}$ on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\widetilde{i}$ is smooth and one has a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,$$

are (-2)-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since π is a double cover with branch locus $B' + \sum_1^t A_i$, it is determined by a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

It is well known that (*cf.* [BPV, Chapter V, Section 22]):

$$
\begin{aligned}
p_g(S) &= p_g(V) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)),\\
q(S) &= q(V) = q(W) + h^1(W, \mathcal{O}_W(K_W + L))
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
K_S^2 - t &= K_V^2 = 2(K_W + L)^2,\\
\chi(\mathcal{O}_S) &= \chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + \tfrac{1}{2}L(K_W + L).
\end{aligned}
\tag{2.2}
$$

Denote by ϕ_2 the bicanonical map of S. From the papers [CM2] and [CbCM]:

$$\phi_2 \text{ is composed with } i \text{ if and only if } h^0(W, \mathcal{O}_W(2K_W + L)) = 0. \tag{2.3}$$

Let P be a minimal model of the resolution W of S/i and $\rho : W \to P$ be the corresponding projection. Denote by $\overline{B}$ the projection $\rho(B)$ and by δ the "projection" of L.

Remark 2.1.1 *If $\overline{B}$ is singular, there are exceptional divisors E_i and numbers $r_i \in 2\mathbb{N}$ such that*

$$
\begin{aligned}
&E_i^2 = -1,\\
&K_W \equiv \rho^*(K_P) + \textstyle\sum E_i,\\
&2L \equiv B = \rho^*(\overline{B}) - \textstyle\sum r_i E_i \equiv \rho^*(2\delta) - \textstyle\sum r_i E_i.
\end{aligned}
$$

Proposition 2.1.2 *With the previous notation, if S is a surface of general type then:*

a) $\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2) - h^0(W, \mathcal{O}_W(2K_W + L))$;

b) $\delta^2 = -2\chi(\mathcal{O}_P) - 2K_P^2 - 3K_P\delta +$
$+\frac{1}{4}\sum(r_i - 2)(r_i - 4) + 2h^0(W, \mathcal{O}_W(2K_W + L))$.

Proposition 2.1.3 *Let t be the number of nodes of S/i. One has:*

a) $t = K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L))$;

b) $t = K_S R'' + 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S) \geq 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S)$;

c) $K_S^2 \geq 2\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) + 2h^0(W, \mathcal{O}_W(2K_W + L))$.

Proposition 2.1.4 *With the above notation:*

a) $h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{3}K_W^2 - \chi(\mathcal{O}_W) + \frac{11}{3}\chi(\mathcal{O}_S) + \frac{1}{27}K_S^2$;

b) $h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2}K_W^2 + 5\chi(\mathcal{O}_S) + 2q(S) - 3\chi(\mathcal{O}_W) - 2q(W)$.

Proof of Proposition 2.1.2: (*cf.* [CM2])

a) From the Kawamata-Viehweg's vanishing theorem (see *e.g.* [EV, Corollary 5.12, c)]), one has

$$h^i(W, \mathcal{O}_W(2K_W + L)) = 0, \ i = 1, 2.$$

The Riemann-Roch theorem implies

$$\chi(\mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L) + K_W(K_W + L),$$

thus, using (2.2),

$$h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_W) + K_W(K_W + L). \tag{2.4}$$

With the notation of Remark 2.1.1, we can write

$$\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = \frac{1}{2}K_W(2K_W + 2L) - h^0(W, \mathcal{O}_W(2K_W + L)) =$$

$$= \frac{1}{2}\left(\rho^*(K_P) + \sum E_i\right)\left(2\rho^*(K_P + \delta) + \sum(2 - r_i)E_i\right) - h^0(W, \mathcal{O}_W(2K_W + L)) =$$

$$= K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2) - h^0(W, \mathcal{O}_W(2K_W + L)).$$

b) From the proof of a),

$$h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_W) + \frac{1}{2}(2K_W + L)(K_W + L).$$

Using Remark 2.1.1 this means

$$h^0(W, \mathcal{O}_W(2K_W + L)) = \chi(\mathcal{O}_P) +$$

$$+\frac{1}{2}\left(\rho^*(2K_P + \delta) + \frac{1}{2}\sum(4 - r_i)E_i\right)\left(\rho^*(K_P + \delta) + \frac{1}{2}\sum(2 - r_i)E_i\right) =$$

$$= \chi(\mathcal{O}_P) + K_P^2 + \frac{3}{2}K_P\delta + \frac{1}{2}\delta^2 - \frac{1}{8}\sum(r_i - 2)(r_i - 4).$$

$\square$

Proof of Proposition 2.1.3:

a) From formulas (2.2) and (2.4),

$$t = K_S^2 - 2K_W(K_W+L) - 2L(K_W+L) =$$

$$= K_S^2 + 2\chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_W) - 2h^0(W, \mathcal{O}_W(2K_W+L)) - 4\chi(\mathcal{O}_S) + 8\chi(\mathcal{O}_W).$$

b) (This is also a consequence of the holomorphic fixed point formula.)
From (2.2) and a),

$$4\chi(\mathcal{O}_S) - 8\chi(\mathcal{O}_W) = 2L(K_W+L) = \left(B' + \sum_1^t A_i\right)(K_W+L) =$$

$$= B'(K_W+L) - t = \frac{1}{2}\pi^*(B')\pi^*(K_W+L) - t = R''K_S - t.$$

Since S is of general type, $K_S R'' \geq 0$, thus

$$t \geq 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S).$$

c) This is immediate from a) and b). □

Proof of Proposition 2.1.4:

a) This inequality is given by the following three claims.

Claim 1:
$1 - p_a(B') = 3\chi(\mathcal{O}_W) - 3\chi(\mathcal{O}_S) - K_S^2 - K_W^2 + 3h^0(W, \mathcal{O}_W(2K_W+L)).$

Proof : Formulas (2.2) and (2.4) give

$$L^2 - K_W^2 =$$

$$= [2\chi(\mathcal{O}_S) - 4\chi(\mathcal{O}_W) - LK_W] - [h^0(W, \mathcal{O}_W(2K_W+L)) - \chi(\mathcal{O}_S) + \chi(\mathcal{O}_W) - K_W L],$$

thus

$$L^2 = K_W^2 + 3\chi(\mathcal{O}_S) - 5\chi(\mathcal{O}_W) - h^0(W, \mathcal{O}_W(2K_W+L)). \tag{2.5}$$

Now we perform a straightforward calculation using the adjunction formula, (2.2), Proposition 2.1.3, a) and (2.5):

$$2p_a(B') - 2 =$$

$$= K_W B' + B'^2 = K_W 2L + (2L)^2 + 2t = 2L(K_W + L) + 2t + 2L^2 =$$

$$2[2\chi(\mathcal{O}_S) - 4\chi(\mathcal{O}_W)] +$$

$$+2[K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L))] +$$

$$+2[K_W^2 + 3\chi(\mathcal{O}_S) - 5\chi(\mathcal{O}_W) - h^0(W, \mathcal{O}_W(2K_W + L))] =$$

$$= 2K_S^2 + 2K_W^2 + 6\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_W) - 6h^0(W, \mathcal{O}_W(2K_W + L)). \diamondsuit$$

Denote by τ the number of rational curves of B'.
Claim 2:

$$1 - p_a(B') \leq \tau.$$

Proof : Write

$$B' = \sum_1^\tau B_i' + \sum_{\tau+1}^h B_i'$$

as a decomposition of B' in (smooth) connected components such that B_i', $i \leq \tau$, are the rational ones. The adjunction formula gives

$$2p_a(B') - 2 = \sum_1^h \left(K_W B_i' + B_i'^2\right) = \sum_1^\tau (2g(B_i') - 2) + \sum_{\tau+1}^h (2g(B_i') - 2) \geq -2\tau. \diamondsuit$$

Claim 3:

$$\tau \leq 8\left(\chi(\mathcal{O}_S) - \frac{1}{9}K_S^2\right).$$

Proof : Since B' does not contain (-2)-curves and it is contained in the branch locus of the cover $\pi : V \to W$, then each rational curve in B' corresponds to a rational curve in S. Now the result follows from Proposition 1.1.5. $\diamondsuit$

Therefore $1 - p_a(B') \le 8\left(\chi(\mathcal{O}_S) - \frac{1}{9}K_S^2\right)$ and using Claim 1 we obtain the desired inequality.

b) Proposition 2.1.3, a) says that

$$K_V^2 = K_S^2 - t = 2\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_W) + 2h^0(W, \mathcal{O}_W(2K_W + L)).$$

The second Betti number b_2 of a surface X satisfies

$$b_2(X) = 12\chi(\mathcal{O}_X) - K_X^2 + 4q(X) - 2.$$

Therefore

$$b_2(V) = 10\chi(\mathcal{O}_V) + 6\chi(\mathcal{O}_W) + 4q(V) - 2 - 2h^0(W, \mathcal{O}_W(2K_W + L)).$$

Since $b_2(V) \ge b_2(W)$, one has the result. □

2.2 Surfaces with an involution and $q = 1$

Let S be a surface of general type with $q = 1$. Then the Albanese variety of S is an elliptic curve E and the Albanese map is a connected fibration (see *e.g.* [Be] or [BPV]).

Suppose that S has an involution i. Then i preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{h} & S & \longrightarrow & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ W & \longrightarrow & S/i & \longrightarrow & \Delta \end{array} \tag{2.6}$$

where Δ is a curve of genus ≤ 1. Denote by

$$f_A : W \to \Delta$$

the fibration induced by the Albanese fibration of S.

Recall that

$$\rho : W \to P$$

is the projection of W onto its minimal model P and

$$\overline{B} := \rho(B),$$

where $B := B' + \sum_1^t A_i \subset W$ is the branch locus of π. Let

$$\overline{B'} := \rho(B'), \quad \overline{A_i} = \rho(A_i).$$

When $\overline{B}$ has only negligible singularities the map ρ contracts only exceptional curves contained in fibres of f_A. In fact otherwise there exists a (-1)-curve $J \subset W$ such that $JB = 2$ and $\pi^*(J)$ is transverse to the fibres of the (genus 1 base) Albanese fibration of S. This is impossible because $\pi^*(J)$ is a rational curve. Moreover ρ contracts no curve meeting $\sum A_i$, thus the singularities of $\overline{B}$ are exactly the singularities of $\overline{B'}$, *i.e.* $\overline{B'} \bigcap \sum \overline{A_i} = \emptyset$. We denote the image of f_A on P by $\overline{f_A}$.

If $\Delta \cong \mathbb{P}^1$ then the double cover $E \to \Delta$ is ramified over 4 points p_j of Δ, thus the branch locus $B' + \sum_1^t A_i$ is contained in 4 fibres

$$F_A^j := f_A^*(p_j), \ j = 1, ..., 4,$$

of the fibration f_A. Hence by Zariski's Lemma (see *e.g.* [BPV]) the irreducible components B_i' of B' satisfy $B_i'^2 \leq 0$. If $\overline{B}$ has only negligible singularities then also $\overline{B'}^2 \leq 0$. Since $\pi^*(F_A^j)$ has even multiplicity, each component of F_A^j which is not a component of the branch locus $B' + \sum_1^t A_i$ must be of even multiplicity.

Chapter 3

ϕ_2 composed with i and $\mathrm{Kod}(S/i) \geq 0$

3.1 S regular

Assume that S is a smooth minimal surface of general type, with $q = 0$ and $p_g \neq 0$, having an involution i such that S/i is non-ruled and the bicanonical map ϕ_2 of S is composed with i. Thus, according to the following Proposition, S/i is birational to a $K3$ surface, $p_g(S) = 1$ and $K_S^2 \leq 8$.

We keep the notation of Chapter 2.

Proposition 3.1.1 (*cf.* also [Xi2]) *Let S be a surface of general type, with $q = 0$ and $p_g \geq 1$, having an involution i such that the bicanonical map ϕ_2 of S is composed with i.*

Then S/i is a rational surface, except in the case $p_g(S) = 1$, where S/i may also be birational to a $K3$ surface. In this last case the branch locus $\overline{B}$ has at most negligible singularities and $K_S^2 \leq 8$.

Proof: Since $q(P) \leq q(S) = 0$ and $p_g(P) \leq p_g(S)$, then $\chi(O_P) - \chi(O_S) \leq 0$. As ϕ_2 is composed with i, $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$. Now Proposition 2.1.2, a) excludes the possibility $\mathrm{Kod}(P) = 2$, because in that case $K_P(K_P + \delta) > 0$.

Suppose $\mathrm{Kod}(P) = 1$. Proposition 2.1.2, a) implies $K_P^2 = K_P\delta = 0$, but then $\overline{B}$ is contained in the elliptic fibration of P, and so also S has an elliptic fibration, a contradiction because S is of general type.

From the classification of surfaces (see *e.g.* [Be, Théorème VIII.2]) and Proposition 2.1.2, a) we obtain P a $K3$ surface as the only possibility for $\mathrm{Kod}(P) = 0$.

In this case $\sum(r_i - 2) = 0$ (meaning that $\overline{B}$ has at most negligible singularities), therefore, since the number of nodal curves in a $K3$ surface is not greater than 16, Proposition 2.1.3 gives $K_S^2 \leq 8$. □

The main result of this section is

Theorem 3.1.2 *Let S be a smooth minimal regular surface of general type with an involution i such that ϕ_2 is composed with i and that S/i is birational to a $K3$ surface. Let W be the minimal resolution of S/i.*

Then W is a double plane and there is a plane model of W with branch locus the union of two cubics.

Let S be one of the surfaces constructed in [To2], as described in Section 1.1. Todorov claimed that the bicanonical map ϕ_2 of S has degree 2 but, as noted in [CD], this is not true if $K_S^2 = 2$: the set $\{A_7, \ldots, A_{16}\}$ may contain a subset of 8 curves A_i with sum divisible by 2 in the Picard group. In this case S has torsion and then ϕ_2 is of degree 4 onto a quadric cone (see [CD]).

One can choose the six nodes in such a way to obtain S with $K^2 = 2$, $p_g = 1$, $q = 0$ and bicanonical map of degree 2 onto a quartic $K3$. Moreover, imposing the passage of the branch curve by a 7-th node, one can obtain S with $K^2 = p_g = 1$ and $q = 0$. In this case ϕ_2 is of degree 4 onto $\mathbb{P}^2$. This is the so-called *Kunev surface.* It is a bidouble cover of $\mathbb{P}^2$ ramified over a general line and two cubics (*cf.* [To1]).

Since the Todorov construction involves a Kummer surface, a natural question arises: is there a surface S of general type with $p_g = 1$, $q = 0$ and $K^2 \geq 2$ having an involution i such that S/i is birational to a $K3$ and non-birational to a Kummer surface? The answer is yes, we give examples in Theorem 3.1.5.

In order to prove Theorem 3.1.2, we show the following:

Proposition 3.1.3 *Let P be a smooth $K3$ surface with a reduced curve B satisfying:*

(i) *$B = B' + \sum_1^t A_i$, $t \in \{9, \ldots, 16\}$, where B' is a nef and big curve with at most negligible singularities, the curves A_i are disjoint (-2)-curves also disjoint from B' and $B \equiv 2L$, $L^2 = -4$, for some $L \in \mathrm{Pic}(P)$.*

Then:

a) *Let $\pi : V \to P$ be a double cover with branch locus B and S be the smooth minimal model of V. Then $q(S) = 0$, $p_g(S) = 1$, $K_S^2 = t - 8$ and ϕ_2 is composed with the involution i of S induced by π;*

b) *If $t \geq 10$, then P contains a smooth curve B'_0 and (-2)-curves $A'_1, \ldots, A'_{t-1}$ such that $B'^2_0 = B'^2 - 2$ and $B_0 := B'_0 + \sum_1^{t-1} A'_i$ also satisfies condition (i).*

Proof:

a) From the double cover formulas and the Riemann Roch theorem,

$$q(S) = h^1(P, \mathcal{O}_P(L)),$$

$$p_g(S) = 1 + h^0(P, \mathcal{O}_P(L)),$$

$$h^0(P, \mathcal{O}_P(L)) + h^0(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(L)).$$

Since $2L - \sum A_i$ is nef and big, the Kawamata-Viehweg's vanishing theorem (see *e.g.* [EV, Corollary 5.12, c)]) implies

$$h^1(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(K_P + L)) = h^1(P, \mathcal{O}_P(L)) = 0,$$

hence $q(S) = 0$ and $p_g(S) = 1$. As

$$h^0(P, \mathcal{O}_P(2K_P + L)) = h^0(P, \mathcal{O}_P(L)) = 0,$$

the bicanonical map of S is composed with i.

b) Denote by $\xi \subset P$ the set of irreducible curves which do not intersect B' and denote by ξ_i, $i \geq 1$, the connected components of ξ. Since $B'^2 \geq 2$, the Hodge-index theorem implies that the intersection matrix of the components of ξ_i is negative definite. Therefore, following [BPV, Lemma I.2.12], the ξ_i's have one of the five configurations: the support of A_n, D_n, E_6, E_7 or E_8 (see *e.g.* [BPV, III.3] for the description of these graphs).

Claim 1: *Each nodal curve A_i can only be contained in a graph of type A_{2n+1} or D_n.*

Proof : Suppose that there exists an A_i which is contained in a graph of type E_6. Denote the components of E_6 as in Figure 3.1.

If $A_i = a_3$ or $A_i = a_6$, then $a_6 B = a_6 a_3 = 1$ or $a_3 B = 1$, contradicting $B \equiv 2L$. If $A_i = a_1$ or $A_i = a_2$, then $a_2 B = 1$ or $a_1 B = 1$, the same contradiction. By the same reason, $A_i \neq a_4$ and $A_i \neq a_5$.

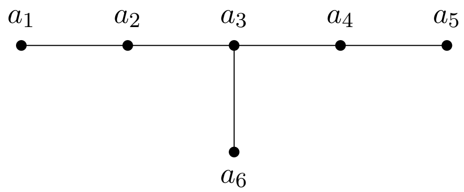


Figure 3.1: E_6

Analogously one can verify that each A_i can not be in a graph of type A_{2n}, E_7 or E_8. $\diamondsuit$

Fix one of the curves A_i and denote by G the graph containing it. The possible configurations for the curves A_i in the graphs are shown in Figure 3.2.

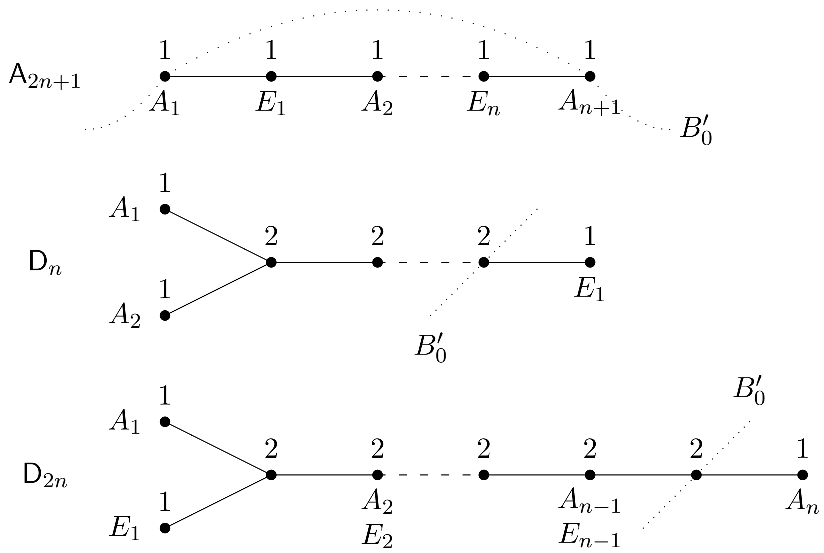


Figure 3.2: *The numbers represent the multiplicity and the doted curve re- present a general element* B'_0 *in* $|B' - \mathsf{G}|$.

Claim 2: *We can choose* A_i *such that the linear system* $|B' - \mathsf{G}|$ *has no fixed components (and thus no base points, from* [St, Theorem 3.1]*).*

Proof : Denote by $\varphi_{|B'|}$ the map given by the linear system $|B'|$. We know that $\varphi_{|B'|}$ is birational or it is of degree 2 (see [St, Section 4]). If $\varphi_{|B'|}$ is birational or the point $\varphi_{|B'|}(\mathsf{G})$ is a smooth point of $\varphi_{|B'|}(P)$, the result is clear, since $|B' - \mathsf{G}|$

is the pullback of the linear system of the hyperplanes containing $\varphi_{|B'|}(\mathsf{G})$ and $\varphi^*_{|B'|}(\varphi_{|B'|}(\mathsf{G})) = \mathsf{G}$ (see [BPV, Theorems III 7.1 and 7.3]).

Suppose now that $\varphi_{|B'|}$ is non-birational and that $\varphi_{|B'|}(\mathsf{G})$ is a singular point of $\varphi_{|B'|}(P)$. Then B' is linearly equivalent to a curve with one of the configurations described in Theorem 1.1.3. Except for the last configuration, G contains at most two (-2)-curves. But $t \geq 9$, thus in these cases there exists other graph G' containing a curve A_j such that $\varphi_{|B'|}(\mathsf{G}')$ is a non-singular point of $\varphi_{|B'|}(P)$ (notice that Theorem 1.1.3 implies that $\varphi_{|B'|}(P)$ contains only one singular point).

So we can suppose that B' is equivalent to a curve with a configuration as in Theorem 1.1.3, (iii), b). None of the curves $\Gamma_0, \ldots, \Gamma_N$ can be one of the curves A_j. For this note that: if $\Gamma_0 = A_j$, then $EB = E\,(B' + \sum A_i) = 2 + E\Gamma_0 = 3 \not\equiv 0 \pmod 2$; if $\Gamma_1 = A_j$, then $\Gamma_0 B = \Gamma_0\Gamma_1 = 1 \not\equiv 0 \pmod 2$; etc. Again this configuration can contain at most two curves A_j, the components Γ_{N+1}, Γ_{N+2}. $\diamondsuit$

Let B_0' be a smooth curve in $|B' - \mathsf{G}|$. If G is an A_{2n+1} graph, then, using the notation of Figure 3.2,

$$\left(B_0' + \sum_1^n E_i\right) + \sum_{n+2}^t A_i \equiv \left(B' - \sum_1^{n+1} A_i\right) + \sum_{n+2}^t A_i \equiv$$

$$\equiv B' + \sum_1^t A_i - 2\sum_1^{n+1} A_i \equiv 0 \pmod 2.$$

Therefore the curve

$$B_0 := B_0' + \sum_1^n E_i + \sum_{n+2}^t A_i$$

satisfies condition (i).

The case where G is a D_m graph is analogous. $\square$

Proof of Theorem 3.1.2: Let P be the minimal model of W. As noted in Proposition 3.1.1, the branch locus $\overline{B} \subset P$ has at most negligible singularities, thus, from Propositions 2.1.2 and 2.1.3, it satisfies condition (i) of Proposition 3.1.3. Then P contains a curve B_0' and (-2)-curves A_i', $i = 1, \ldots, 9$, such that $B_0 := B_0' + \sum_1^9 A_i'$ is smooth and divisible by 2 in the Picard group. Moreover, the complete linear system $|B_0'|$ has no fixed component nor base points and $B_0'^2 = 2$. Therefore, from [St], $|B_0'|$ defines a generically finite degree 2 morphism

$$\varphi := \varphi_{|B_0'|} : P \to \mathbb{P}^2.$$

Since $g(B'_0) = 2$, this map is ramified over a sextic curve β. The singularities of β are negligible because P is a $K3$ surface.

We claim that β is the union of two cubics. Let $p_i \in \beta$ be the singular point corresponding to A'_i, $i = 1, \ldots, 9$. Notice that the p_i's are possibly infinitely near. Let $C \subset \mathbb{P}^2$ be a cubic curve passing through p_i, $i = 1, \ldots, 9$. As $C + \varphi_*(B'_0)$ is linearly equivalent to a plane quartic, we have

$$\left(\varphi^*(C) - \sum_1^9 A'_i\right) + B'_0 + \sum_1^9 A'_i \equiv \varphi^*(C + \varphi_*(B'_0)) \equiv 0 \pmod 2,$$

hence also $\varphi^*(C) - \sum_1^9 A'_i \equiv 0 \pmod 2$, *i.e.* there exists a divisor J such that

$$2J \equiv \varphi^*(C) - \sum_1^9 A'_i.$$

Since P is a K_3 surface, the Riemann Roch theorem implies that J is effective. From $JA'_i = 1$, $i = 1, \ldots, 9$, we obtain that the plane curve $\varphi_*(J)$ passes with multiplicity 1 through the nine singular points p_i of β. This immediately implies that $\varphi_*(J)$ is not a line nor a conic, because β is a reduced sextic. Therefore $\varphi_*(J)$ is a reduced cubic. So $\varphi_*(J) \equiv C$ and then

$$\varphi^*\left(\varphi_*(J)\right) \equiv 2J + \sum_1^9 A'_i.$$

This implies that $\varphi_*(J)$ is contained in the branch locus β. □

As an example, we describe briefly how to construct a surface S of general type such that $\phi_2(S) \subset \mathbb{P}^3$ is a quartic $K3$ surface with an A_{17} and A_1 singularities. The details can be verified with Magma.

Example 3.1.4 *Let C_1 be a nodal cubic, p an inflection point of C_1 and T the tangent line to C_1 at p. The pencil generated by C_1 and $3T$ contains another nodal cubic C_2, smooth at p. The curves C_1 and C_2 intersect at p with multiplicity 9.*

Let $\rho : X \to \mathbb{P}^2$ be the resolution of C_1+C_2 and $\pi : W \to X$ be the double cover with branch locus the strict transform of $C_1 + C_2$. Denote by $\bar{l}$ the line containing the nodes of C_1 and C_2 and by $l \subset W$ the pullback of the strict transform of $\bar{l}$. The map given by $|(\rho \circ \pi)^*(\bar{l}) + l|$ is birational onto a quartic Q in $\mathbb{P}^3$ with an A_1 and A_{17} singularities (notice that l is a (-2)-curve and $((\rho \circ \pi)^*(\bar{l}) + l)l = 0$).

Let $B' \in |(\rho \circ \pi)^*(\bar{l}) + l|$ be a smooth element and $A_1, \ldots, A_9$ be the disjoint (-2)-curves contained in $(\rho \circ \pi)^*(p)$. Let S be the minimal model of the double *cover of W with branch locus $B' + \sum_1^9 A_i + l$. The surface Q is the image of the bicanonical map of S and $p_g(S) = 1$, $q(S) = 0$, $K_S^2 = 2$.*

Finally, we want to know if there are examples of surfaces S satisfying the assumptions of Theorem 3.1.2 and not in the family constructed by Todorov.

It is known since [Hu] that there exist special sets of 6 nodes, called Weber hexads, in the Kummer surface $Q \in \mathbb{P}^3$ such that the surface which is the blow-up of Q at these nodes can be embedded in $\mathbb{P}^3$ as a quartic with 10 nodes. This quartic is the Hessian of a smooth cubic surface.

The space of all smooth cubic surfaces has dimension 4 while the space of Kummer surfaces has dimension 3. Thus it is natural to ask if there exist Hessian "non-Kummer" surfaces, *i.e.* which are not the embedding of a Kummer surface blown-up at 6 points. This is studied in [Ro], where the existence of "non-Kummer" quartic Hessians H in $\mathbb{P}^3$ is shown. These are surfaces with 10 nodes a_i such that the projection from one node a_1 to $\mathbb{P}^2$ is a generically $2:1$ cover of $\mathbb{P}^2$ with branch locus $\alpha_1 + \alpha_2$ satisfying: α_1, α_2 are smooth cubics tangent to a nondegenerate conic C at 3 distinct points. We use this in the next result.

Theorem 3.1.5 *There exist minimal smooth surfaces S of general type with $K^2 = 2, 3$, $p_g = 1$, $q = 0$ and bicanonical map of degree 2 onto a surface which is birational to a $K3$ and non-birational to a Kummer surface.*

Proof: Let α_1, α_2 and C be as above. Take the morphism $\pi : W \to \mathbb{P}^2$ given by the canonical resolution of the double cover of $\mathbb{P}^2$ with branch locus $\alpha_1 + \alpha_2$. The strict transform of C gives rise to the union of two disjoint (-2)-curves $A_1, A_2 \subset W$ (one of these correspond to the node a_1 from which we projected).

Let $T \in \mathbb{P}^2$ be a general line. Let $A_3, \ldots, A_{11} \subset W$ be the disjoint (-2)-curves contained in $\pi^*(\alpha_1 + \alpha_2)$. We have $\pi^*(T + \alpha_1) \equiv 0 \pmod 2$, hence, since α_1 is in the branch locus, also

$$\pi^*(T) + \sum_3^{11} A_i \equiv 0 \pmod 2.$$

The linear systems $|\pi^*(T) + A_2|$ and $|\pi^*(T) + A_1 + A_2|$ have no fixed components nor base points (see [St, (2.7.3) and Corollary 3.2]). The surface S is the minimal model of the double cover of W ramified over a general element in

$$|\pi^*(T) + A_2| + \sum_2^{11} A_i \quad \text{or} \quad |\pi^*(T) + A_1 + A_2| + \sum_1^{11} A_i.$$

□

3.2 S irregular

3.2.1 Classification theorem

In this section we use freely the notation and results of Chapter 2, and we prove the following:

Theorem 3.2.1 *Let S be a smooth minimal irregular surface of general type with an involution i such that $\mathrm{Kod}(S/i) \geq 0$ and the bicanonical map ϕ_2 of S is composed with i. If $p_g(S) = q(S) = 1$, then only the following possibilities can occur:*

a) *S/i is regular, the Albanese fibration of S has genus 2 and*

  (i) $\mathrm{Kod}(S/i) = 2$, $\chi(S/i) = 2$, $K_S^2 = 2$, $\deg(\phi_2) = 8$, *or*

  (ii) $\mathrm{Kod}(S/i) = 1$, $\chi(S/i) = 2$, $2 \leq K_S^2 \leq 4$, $\deg(\phi_2) \geq 4$, *or*

  (iii) *S/i is birational to a $K3$ surface,* $3 \leq K_S^2 \leq 6$, $\deg(\phi_2) = 4$;

b) *S has no genus 2 fibration and S/i is birational to a $K3$ surface.*

Moreover, there are examples for (i), (ii) *with* $K_S^2 = 4$, (iii) *with* $K_S^2 = 3, 4$ *or* 5 *and for* b) *with* $K_S^2 = 6$ *and* ϕ_2 *of degree* 2.

Remark 3.2.2 *Examples for* (iii) *were given by Catanese in* [Ca2]. *The other examples will be presented in Sections* 3.2.2, 6.1 *and* 6.2.

Proof: Since $p_g(P) \leq p_g(S) = 1$, then $\chi(\mathcal{O}_P) \leq 2 - q(P) \leq 2$. Proposition 2.1.2 gives $\chi(\mathcal{O}_P) \geq 1$, because K_P is nef (*i.e.* $K_P C \geq 0$ for every curve C). So from Proposition 2.1.2 and the classification of surfaces (see *e.g.* [Be] or [BPV]) only the following cases can occur:

1) P is a surface of general type;

2) P is a surface with Kodaira dimension 1;

3) P is an Enriques surface, $\overline{B}$ has only negligible singularities;

4) P is a K3 surface, $\overline{B}$ has a 4-uple or $(3,3)$-point, and possibly negligible singularities.

We will show that case 3) does not occur and that in cases 1) and 2) the Albanese fibration has genus 2.

Each of cases $1), \ldots, 4)$ will be studied separately. We start by consi- dering:

Case 1) Since P is of general type, $K_P^2 \geq 1$ and K_P is nef, Proposition 2.1.2 gives $\chi(\mathcal{O}_P) = 2$, $K_P^2 = 1$, $K_P\delta = 0$ and $\overline{B}$ has only negligible singularities. The equality $K_P\overline{B'} = K_P 2\delta = 0$ implies $\overline{B'}^2 < 0$ when $B' \neq 0$. In the notation of Remark 2.1.1 one has $K_W \equiv \rho^*(K_P) + \sum E_i$ and $B' = \rho^*(\overline{B'}) - 2\sum E_i$. So

$$K_S^2 = K_V^2 + t = \frac{1}{4}(2K_V)^2 + t = \frac{1}{4}\pi^*(2K_W + B)^2 + t =$$

$$= \frac{1}{2}(2K_W + B)^2 + t = \frac{1}{2}(2K_W + B')^2 = \frac{1}{2}(2K_P + \overline{B'})^2 = \frac{1}{2}(4 + \overline{B'}^2).$$

Since $K_S^2 \geq 2p_g(S)$ for an irregular surface (see [De]), $\overline{B'}^2 < 0$ is impossible, hence $B' = 0$ and $K_S^2 = 2$. By [Ca1], minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 2$ have Albanese fibration of genus 2. This is case (i) of Theorem 3.2.1. We will see in Section 6.2 an example for this case.

Finally the fact that $\deg(\phi_2) = 8$ follows immediately because ϕ_2 is a morphism onto $\mathbb{P}^2$ and $(2K_S)^2 = 8$.

Next we exclude:

Case 3) Using the notation of Remark 2.1.1 of Section 2.1, we write

$$K_W \equiv \rho^*(K_P) + \sum E_i \quad \text{and} \quad 2L \equiv \rho^*(2\delta) - 2\sum E_i,$$

for some exceptional divisors E_i. Hence

$$L(K_W + L) = \frac{1}{2}L(2K_W + 2L) =$$

$$= \frac{1}{2}(\rho^*(\delta) - \sum E_i)(2\rho^*(K_P) + \rho^*(2\delta)) = \frac{1}{2}\delta(2K_P + 2\delta) = \delta^2$$

and then, from (2.2), $\delta^2 = -2$. Now (2.3) and Proposition 2.1.3, a) imply $t = K_S^2 + 4$, thus

$$\overline{B'}^2 = \overline{B}^2 + 2t = (2\delta)^2 + 2t = -8 + 2t = 2K_S^2 > 0.$$

This is a contradiction because we have seen in Section 2.2 that $\overline{B'}^2 \leq 0$ when $\overline{B'}$ has only negligible singularities. Thus case 3) does not occur.

Now we focus on:

Case 2) Since we are assuming that $\mathrm{Kod}(P) = 1$, P has an elliptic fibration, *i.e.* a morphism $f_e : P \to C$ where C is a curve and the general fibre of f_e is a smooth connected elliptic curve. Then K_P is numerically equivalent to a rational multiple of a fibre of f_e (see *e.g.* [Be] or [BPV]). As $K_P\delta \geq 0$, Proposition 2.1.2, together with $\chi(\mathcal{O}_P) \leq 2$, yield $K_P\delta = 0$ or 1.

Denote by F_e a general fibre of f_e. If $K_P\delta = 0$, then $F_e\overline{B} = 0$, which implies that the fibration f_e lifts to an elliptic fibration on S. This is impossible because S is a surface of general type. So $K_P\delta = 1$ and, since $p_g(P) \leq p_g(S) = 1$, the only possibility allowed by Proposition 2.1.2 is

$p_g(P) = 1$, $q(P) = 0$ and $\overline{B}$ has only negligible singularities.

Now $q(P) = 0$ implies that the elliptic fibration f_e has a rational base, thus the canonical bundle formula (see *e.g.* [BPV, Chapter V, Section 12]) gives $K_P \equiv \sum(m_i - 1)F_i$, where m_iF_i are the multiple fibres of f_e. From

$$2 = 2\delta K_P = \overline{B'}K_P = \overline{B'}\sum(m_i - 1)F_i, \qquad \overline{B'}F_i \equiv 0 \pmod 2)$$

we get

$$K_P \equiv \frac{1}{2}F_e.$$

Since $\overline{B}$ has only negligible singularities, $\overline{B'}^2 \leq 0$ and

$$2K_S^2 = (2K_W + B')^2 = \rho^*\left(2K_P + \overline{B'}\right)^2 = 8 + \overline{B'}^2 \leq 8. \tag{3.1}$$

Therefore $2 \leq K_S^2 \leq 4$. If $K_S^2 = 2$, then the Albanese fibration of S is of genus 2, by [Ca1]. So, to prove statement a), (ii) of Theorem 3.2.1, we must show that for $K_S^2 = 3$ or 4 the Albanese fibration of S has genus 2. We will study each of these cases separately.

Consider the fibration $f_A : W \to \mathbb{P}^1$, with general fibre F_A, induced by the Albanese fibration of S. Recall from Section 2.2 that the branch locus of $\pi : V \to W$ is contained in four fibres F_A^i, $i = 1, \ldots, 4$, of f_A. Denote the images of f_A, F_A, F_A^i on the minimal model P of W by $\overline{f_A}$, $\overline{F_A}$, $\overline{F_A^i}$, respectively.

Suppose that

· $\mathbf{K_S^2 = 4}$.

Claim 1: *If f_A is not a genus 2 fibration then*

$$\overline{F_A^j} = 2\overline{B'},$$

for some $j \in \{1, \ldots, 4\}$.

Proof : By formula (3.1) $\overline{B'}^2 = 0$, and so $\overline{B'}$ contains the support of $x \geq 1$ of the $\overline{F_A^i}$'s. The facts $K_P\overline{F_A} > 0$ (because $g(\overline{F_A}) \geq 2$) and $K_P\overline{B'} = 2$ imply $x = 1$, *i.e.* $\overline{F_A^j} = k\overline{B'}$, for some $j \in \{1, \ldots, 4\}$ and $k \in \mathbb{N}$. If $k = 1$ then $\overline{F_A}K_P = 2$, thus $\overline{F_A}$ is of genus 2 and S is as in case (ii) of Theorem 3.2.1.

Suppose now $k \geq 2$. Then each irreducible component of the divisor

$$D := \overline{F_A^1} + \ldots + \overline{F_A^4}$$

whose support is not in $\sum_1^{14} \overline{A_i}$ is of multiplicity greater than 1. The fibration $\overline{f_A}$ gives a cover $F_e \to \mathbb{P}^1$ of degree $\overline{F_A}F_e$, for a general fibre F_e of the elliptic fibration f_e. The Hurwitz formula (see *e.g.* [GH]) says that the ramification degree r of this cover is $2\overline{F_A}F_e$. Let $p_1, \ldots, p_n$ be the points in $F_e \cap D$ and α_i be the intersection number of F_e and D at p_i. Of course $F_eD = 4\overline{F_A}F_e = \sum_1^n \alpha_i$ and then $F_e \bigcap \sum \overline{A_i} = \emptyset$ implies $\alpha_i \geq 2$, $i = 1, \ldots, n$. We have

$$2\overline{F_A}F_e = r \geq \sum_1^n (\alpha_i - 1) = \sum_1^n \alpha_i - n = 4\overline{F_A}F_e - n,$$

i.e. $n \geq 2\overline{F_A}F_e$. The only possibility is $n = 2\overline{F_A}F_e$ and $\alpha_i = 2\ \forall i$, which means that every component Γ of D such that $\Gamma F_e \neq 0$ is exactly of multiplicity 2. In particular any irreducible component of $\overline{B'}$ is of multiplicity 2, thus $k = 2$, *i.e.* $\overline{F_A^j} = 2\overline{B'}$. $\diamondsuit$

Claim 2: *There is a smooth rational curve C contained in a fibre F_C of the elliptic fibration f_e, and not contained in fibres of $\overline{f_A}$, such that*

$$m := \widehat{C}\sum_1^t A_i \leq 3, \tag{3.2}$$

where $\widehat{C}$ is the strict transform of C in W.

Proof : Since $\overline{A_i}F_e = \overline{A_i}2K_P = 0$, then each $\overline{A_i}$ is contained in a fibre of f_e, and in particular the elliptic fibration f_e has reducible fibres. Denote by C an irreducible component of a reducible fibre F_C of f_e, by ξ the multiplicity of C in F_C and by $\widehat{C}$ the strict transform of C in W. If the intersection number of C and the support of $F_C - \xi C$ is greater than 3, then, from the configurations of singular fibres of an elliptic fibration (see *e.g.* [BPV, Chapter V, Section 7]), F_C must be of type I_0^*, *i.e.* it has the following configuration: it is the union of four disjoint

(-2)-curves $\theta_1, \ldots, \theta_4$ with a (-2)-curve θ, with multiplicity 2, such that $\theta\theta_i = 1$, $i = 1, \ldots, 4$.

So if $\widehat{C} \sum_1^t A_i > 3$, the fibre F_C containing C is of type I_0^* with $\widehat{C} \sum_1^t A_i = 4$. Since the number of nodes of S/i is $t = K_S^2 + 10 = 14 \not\equiv 0 \pmod 4$, there must be a reducible fibre such that for every component $C \not\subset \sum_1^t \overline{A_i}$, $\widehat{C} \sum_1^t A_i \leq 3$. As $f_e \neq \overline{f_A}$ and the $\overline{A_i}$'s are contained in fibres of f_e and in fibres of $\overline{f_A}$, we can choose C not contained in fibres of $\overline{f_A}$. $\diamondsuit$

Let C be as in Claim 2 and consider the resolution $\widetilde{V} \to V$ of the singularities of $\pi^*(\widehat{C})$. Let $G \subset \widetilde{V}$ be the strict transform of $\pi^*(\widehat{C})$. Notice that G has multiplicity 1, because C transverse to the fibres of $\overline{f_A}$ implies $C \not\subset \overline{B}$. Recall that E denotes the basis of the Albanese fibration of S.

Claim 3: *The Albanese fibration of $\widetilde{V}$ induces a cover $G \to E$ with ramification degree*

$$r := K_{\widetilde{V}} G + G^2.$$

Proof : Let $G_1, \ldots, G_h$ be the connected (hence smooth) components of G. The curve C is not contained in fibres of $\overline{f_A}$, thus G is not contained in fibres of the Albanese fibration of $\widetilde{V}$. This fibration induces a cover $G_i \to E$ with ramification degree, from the Hurwitz formula,

$$r_i = 2g(G_i) - 2 = K_{\widetilde{V}} G_i + G_i^2.$$

This way we have a cover $G \to E$ with ramification degree

$$r = \sum r_i = K_{\widetilde{V}}(G_1 + \cdots + G_h) + \left(G_1^2 + \cdots + G_h^2\right) = K_{\widetilde{V}} G + G^2. \ \diamondsuit$$

We are finally in position to show that $g(F_A) = 2$.

Let $n := \widehat{C} B'$. We have

$$2K_V \pi^*(\widehat{C}) = \pi^* \left(2K_W + B' + \sum A_i\right) \pi^*(\widehat{C}) =$$

$$= 2\left(2K_W + B' + \sum A_i\right) \widehat{C} = 4K_W \widehat{C} + 2\left(B' + \sum A_i\right) \widehat{C} =$$

$$= 4(-2 - \widehat{C}^2) + 2(n + m) = -8 - 2\pi^*(\widehat{C})^2 + 2(n + m),$$

i.e.

$$K_V\pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 = n + m - 4.$$

Suppose that $g(F_A) \neq 2$. Let $\Lambda \subset V$ be the double Albanese fibre induced by $\overline{F_A^j} = 2\overline{B'}$ (as in Claim 1) and $\widetilde{\Lambda} \subset \widetilde{V}$ be the total transform of Λ. From

$$G\widetilde{\Lambda} = \pi^*(\widehat{C})\Lambda \geq \pi^*(\widehat{C})\pi^*(B') = 2n,$$

one has $r \geq n$. Then

$$n + m - 4 = K_V\pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 \geq K_{\widetilde{V}}G + G^2 = r \geq n$$

and so $m \geq 4$, which contradicts Claim 2.

So if $K_S^2 = 4$, then the Albanese fibration of S is of genus 2.

We now consider the possibility

· $\mathbf{K_S^2 = 3}$.

In this case a general Albanese fibre Λ has genus 2 or 3 (see Theorem 1.3.1). Suppose that $g(\Lambda) = 3$. Surfaces with $K^2 = g(\Lambda) = 3$ are studied in detail in [CC1]. There (see also [Ko]) it is shown that the relative canonical map γ, given by $|K + n\Lambda|$ for some n, is a morphism (Theorem 1.3.1, c)).

We know that $K_P\overline{B'} = 2$ and $\overline{B'}^2 = -2$, by (3.1). We have already seen that $\overline{B}$ has only negligible singularities (which means $r_i = 2\ \forall i$, in the notation of Remark 2.1.1) and then ρ contracts no curve meeting $\sum A_i$.

Claim 4: *We have*

$$K_V R' = 1,$$

where R' is the support of $\pi^*(B')$.

Proof :

$$2K_V \cdot 2R' = \pi^*(2K_W + B)\pi^*(B') = 2(2K_W + B)B' =$$

$$= 2(2K_W + B')B' = 2\left(2\rho^*(K_P) + \rho^*(\overline{B'})\right)\left(\rho^*(\overline{B'}) - \sum 2E_i\right) =$$

$$= 2(2K_P + \overline{B'})\overline{B'} = 2(4 - 2) = 4,$$

thus $K_V R' = 1$. $\diamondsuit$

As the map

$$\gamma \circ h : V \longrightarrow \gamma(S)$$

is a birational morphism, $\gamma \circ h(R')$ is a line (plus possibly some isolated points). This way there exists a smooth rational curve $\beta \subset B'$ such that

$$K_V \widetilde{\beta} = 1,$$

where $\widetilde{\beta} \subset R'$ is the support of $\pi^*(\beta)$. The adjunction formula gives $\widetilde{\beta}^2 = -3$, thus $\beta^2 = -6$. Notice that $\widetilde{\beta}$ is the only component of R' which is not contracted by the map $\gamma \circ h$.

Let

$$\alpha := B' - \beta \subset W,$$

$$\overline{\beta} := \rho(\beta),\ \overline{\alpha} := \rho(\alpha) \subset P.$$

When α is non-empty, the support of $\pi^*(\alpha)$ is an union of (-2)-curves, since it is contracted by $\gamma \circ h$. Equivalently α is a disjoint union of (-4)-curves.

Claim 5: *We have*

$$K_W^2 \geq -2.$$

Proof : The second Betti number b_2 of a surface X satisfies

$$b_2(X) = 12\chi(\mathcal{O}_X) - K_X^2 - 2 + 4q(X).$$

The result follows from

$$b_2(V) = b_2(S) + t = 11 + 13 = 24, \qquad b_2(W) = 22 - K_W^2$$

and $b_2(V) \geq b_2(W)$. $\diamondsuit$

From Claim 5, we conclude that the resolution of $\overline{B'}$ blows-up at most two double points, thus

$$B'^2 \geq -2 + 2(-4) = -10 = \beta^2 + (-4).$$

This implies that α is a smooth (-4)-curve when $\alpha \neq 0$.

Claim 6: *Only the following possibilities can occur:*

- $\overline{\beta}$ *has one double point and no other singularity, or*

· $\overline{\alpha}$, $\overline{\beta}$ *are smooth,* $\overline{\alpha}\overline{\beta} = 2$.

Proof : Recall that $B' = \alpha + \beta$ is contained in fibres of f_A and, since $\overline{B'}$ has only negligible singularities, then also $\overline{B'} = \overline{\alpha} + \overline{\beta}$ is contained in fibres of $\overline{f_A}$. In particular $\overline{\alpha}^2, \overline{\beta}^2 \leq 0$.

If $\overline{\alpha}$ is singular, then it has arithmetic genus $p_a(\overline{\alpha}) = 1$ and $\overline{\alpha}^2 = 0$. But then $\overline{\alpha}$ has the same support of a fibre of $\overline{f_A}$, which is a contradiction because $\overline{f_A}$ is not elliptic. Therefore $\overline{\alpha}$ is smooth.

Since $K_P\overline{\alpha} \geq 0$, $K_P\overline{B'} = 2$ implies $K_P\overline{\beta} \leq 2$. We know that β is a smooth rational curve and $\beta^2 = -6$, thus $K_W\beta = 4$. If $\overline{\beta}$ is smooth, then one must have $\overline{\alpha}\overline{\beta} > 1$. From Claim 5 the only possibility in this case is $\overline{\alpha}\overline{\beta} = 2$. If $\overline{\beta}$ is singular, then $\overline{\beta}^2 \leq 0$ implies that $\overline{\beta}$ has one ordinary double point and no other singularity. $\diamondsuit$

Let $D := \overline{\beta}$ if $\overline{\beta}$ is singular. Otherwise let $D := \overline{\alpha} + \overline{\beta}$.

The 2-connected divisor $\widetilde{D} := \frac{1}{2}(\rho \circ \pi)^*(D)$ has arithmetic genus $p_a(\widetilde{D}) = 1$. We know that $(K_V + n\Lambda)\widetilde{D} = 1$ (because $K_V R' = 1$) and that $\widetilde{D}$ contains a component A such that $(K_V + n\Lambda)A = 0$ (because D has at least one negligible singularity). These two facts imply, from Proposition 1.1.4, that the relative canonical map γ has a base point in $\widetilde{D}$. As mentioned above, γ is a morphism, which is a contradiction.

Finally the assertion about $\deg(\phi_2)$ in Case 2): we have proved that S has a genus 2 fibration, so it has an hyperelliptic involution j. The bicanonical map ϕ_2 factors through both i and j, thus $\deg(\phi_2) \geq 4$.

This finishes the proof of case a), (ii) of Theorem 3.2.1.

We end the proof of Theorem 3.2.1 with **Case a), (iii)**: A surface of general type with a genus 2 fibration and $p_g = q = 1$ satisfies $K^2 \leq 6$ (see [Xi1]). Denote by j the map such that $\phi_2 = j \circ i$. The quotient S/i is a $K3$ surface thus, from [St], $\deg(j) \leq 2$. Analogously to Case 2, deg $(\phi_2) \geq 4$, thus $\deg(j) = 2$, $\deg(\phi_2) = 4$ and then $K_S^2 \neq 2$ (see Case 1).

It follows from [Xi1, p. 66] that, if the genus 2 fibration of S has a rational basis, then $K_S^2 = 3$. It is shown in [Po3] that, in these conditions, $\deg(\phi_2) = 2$. We then conclude that the genus 2 fibration of S is the Albanese fibration.

Examples for case a), (iii) with $K_S^2 = 3, 4$ or 5 were given by Catanese in [Ca2].

The existence of the other cases is proved in Sections 3.2.2, 6.1 and 6.2. $\square$

3.2.2 Example with $K^2 = 6$ and $\phi_2(S)$ birational to a $K3$

Here we construct a smooth minimal surface of general type S with $p_g = q = 1$ having an involution i such that the bicanonical map ϕ_2 of S is composed with i and

· $K_S^2 = 6$, $g = 3$, $\deg(\phi_2) = 2$, S/i is birational to a $K3$ surface,

where g denotes the genus of the Albanese fibration of S.

In [To2] Todorov gives the following construction of a surface of general type S with $p_g = 1$, $q = 0$ and $K^2 = 8$. Consider a Kummer surface Q in $\mathbb{P}^3$, *i.e.* a quartic with 16 nodes (ordinary double points) and no other singularity. Let $G \subset Q$ be the intersection of Q with a general quadric, $\widetilde{Q}$ be the minimal resolution of Q and $\widetilde{G} \subset \widetilde{Q}$ be the pullback of G. The surface S is the minimal model of the double cover $\pi : V \to \widetilde{Q}$ ramified over $\widetilde{G} + \sum_1^{16} A_i$, where $A_i \subset \widetilde{Q}$, $i = 1, \ldots, 16$, are the (-2)-curves which contract to the nodes of Q.

It follows from the double cover formulas (*cf.* [BPV, Chapter V, Section 22]) that the imposition of a quadruple point to the branch locus decreases K^2 by 2 and the Euler characteristic χ by 1.

We will see that we can impose a quadruple point to the branch locus of the Todorov construction, thus obtaining a surface S with $K^2 = 6$. In this case I claim that $p_g(S) = q(S) = 1$. In fact, let W be the surface $\widetilde{Q}$ blown-up at the quadruple point, E be the corresponding (-1)-curve, B be the branch locus and L be the line bundle such that $2L \equiv B$. From formula (2.4) in Section 2.1, one has $h^0(W, \mathcal{O}_W(2E + L)) = 0$ (thus the bicanonical map of V factors through π), hence also $h^0(W, \mathcal{O}_W(E + L)) = 0$ and then

$$p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(E + L)) = 1.$$

We will show that $\deg(\phi_2) = 2$, hence $\phi_2(S)$ is a $K3$ surface and so S has no genus 2 fibration.

The construction of the surface is divided on steps. The corresponding Magma computations are in Appendix A.1.

Step 1:
First we need to obtain an equation of a Kummer surface. The Computational Algebra System *Magma* has a direct way to do this, but I prefer to do it using a beautiful construction that I learned from Miles Reid.

We want a quartic surface $Q \in \mathbb{P}^3$ whose singularities are exactly 16 nodes. Projecting from one of the nodes to $\mathbb{P}^2$, one realizes the "Kummer" surface as a double cover

$$\psi : X \longrightarrow \mathbb{P}^2$$

with branch locus the union of 6 lines l_i (see [GH, p. 774]), each one tangent to a conic C (the image of the projection point) at a point p_i. The surface X contains 15 nodes (from the intersection of the lines) and two (-2)-curves (the pullback $\psi^*(C)$) disjoint from these nodes. To obtain a Kummer surface we have just to contract one of these curves.

Denote also by l_i the defining polynomial of each line l_i. An equation for X is $z^2 = l_1 \cdots l_6$ in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, with coordinates (z, x_1, x_2, x_3). We will see that this equation can be written in the form $AB+DE = 0$, where the system $A = B = D = E = 0$ has only the trivial solution and B, E are the defining polynomials of one of the (-2)-curves in $\psi^*(C)$. Now consider the surface X' given by $Bs = D$, $Es = -A$ in the space $\mathbb{P}(3, 1, 1, 1, 1)$ with coordinates (z, s, x_1, x_2, x_3). There is a morphism $X \to X'$ which restricts to an isomorphism

$$X \backslash \{B = E = 0\} \longrightarrow X' \backslash \{[0 : 1 : 0 : 0 : 0]\}$$

and which contracts the curve $\{B = E = 0\}$ to the point $[0 : 1 : 0 : 0 : 0]$. This is an example of *unprojection* (see [Re2]).

The variable z appears isolated in the equations of X', therefore eliminating z we obtain the equation of the Kummer Q in $\mathbb{P}^3$ with variables (s, x_1, x_2, x_3).

Step 2:
We want to find a quadric H such that $H \bigcap Q$ is a reduced curve $\overline{B'}$ having an ordinary quadruple point *pt* as only singularity. Since the computer is not fast enough when working with more than 5 or 6 variables, we first need to think what the most probable case is.

Like we have seen in Section 2.2, the branch locus $B' + \sum_1^{16} A_i$ is contained in 4 fibres $F_A^1, \ldots, F_A^4$ of a fibration f_A of W, where W is the resolution of Q blown-up at *pt* and the curves A_i are the (-2)-curves which contract to the nodes of Q.

Of course we have a quadric intersecting Q at a curve with a quadruple point pt : the tangent space T to Q at pt counted twice. But this one is double, so we need to find an irreducible one (and these two induce f_A), the curve $\overline{B'}$. These curves $2T$ and $\overline{B'}$ are good candidates for $\overline{F_A^1}$ and $\overline{F_A^2}$ (in the notation of Sections 2.2 and 3.2.1). If this configuration exists, then the 16 nodes must be contained in the other two fibres, $\overline{F_A^3}$ and $\overline{F_A^4}$. These fibres are divisible by 2, because $\overline{F_A^1} = 2T$, and are double outside the nodes. Since in a $K3$ surface only 0, 8 or 16 nodes can have sum divisible by 2, it is reasonable to try the following configuration: each of $\overline{F_A^3}$ and $\overline{F_A^4}$ contain 8 nodes with sum divisible by 2 and is double outside the nodes.

It is well known (see *e.g.* [GH]) that the Kummer surface Q has 16 double hyperplane sections T_i such that each one contains 6 nodes of Q and that any two of them intersect in 2 nodes. The sum of the 8 nodes contained in

$$N := (T_1 \cup T_2) \backslash (T1 \cap T_2)$$

is divisible by 2. Magma will give 3 generators h_1, h_2, h_3 for the linear system of quadrics through these nodes.

Step 3:
Now we want to find a quadric H in the form $h_1 + bh_2 + ch_3$, for some b, c (or, less probably, in the form $bh_2 + ch_3$) such that the projection of $H \bigcap Q$ to $\mathbb{P}^2$ (by elimination) is a curve with a quadruple point. To find a quadruple point we just have to impose the annulation of the derivatives up to order 3 and ask Magma to do the rest.

Step 4:
We have to choose one of the solutions, from the previous step, and show that it works, *i.e.* we need to present a reduced curve in Q with self-intersection 16, having a quadruple point and not containing any of the nodes of Q.

Step 5:
Finally, it remains to be shown that the degree of the bicanonical map ϕ_2 is 2. As $(2K_S)^2 = 24$, it suffices to show that $\phi_2(S)$ is of degree 12. Since, in the notation of diagram (2.6), $h^*|2K_S| = \pi^*|2K_W + B'|$, then $\phi_2(S)$ is the image of W via the map $\tau : W \to \phi_2(S)$ given by $|2K_W + B'|$. The projection of this linear system on Q is the linear system of the quadrics whose intersection with Q has a double point at pt. In order to easily write this linear system, we will translate the point pt to

the origin (in affine coordinates). The calculations of Appendix A.1 confirm that $\deg \phi_2(S) = 12$.

Chapter 4

ϕ_2 composed with i and $\mathrm{Kod}(S/i) = -\infty$

Suppose that S is a surface of general type, without a genus 2 fibration, with $p_g = q = 1$ and having an involution i such that S/i is ruled and the bicanonical map ϕ_2 of S is composed with i.

From Theorem 1.2.4, S is a Du Val double plane. But, to my knowledge, the existence of such double planes with $K^2 = 3, \ldots, 7$ has not been shown yet. In Section 4.2 we obtain equations of plane models of Du Val double planes with $p_g = q = 1$ and $K^2 = 2, \ldots, 8$.

4.1 Useful pencils

Here we show the existence of some pencils of plane curves that are useful on some of the constructions of Section 4.2 and Chapter 6.

Notation 4.1.1 *Let $p_0, \ldots, p_j, \ldots, p_{j+s} \in \mathbb{P}^2$ be distinct points and define T_i as the line through p_0 and p_i, $i = 1, \ldots, j$. We say that a plane curve is of type*

$$d(m, (n,n)_T^j, r^s)$$

if it is of degree d and if it has: an m-uple point at p_0, an (n,n)-point at $p_1, \ldots, p_j$, an r-uple point at $p_{j+1}, \ldots, p_{j+s}$ and no other non-negligible singularities. The index $_T$ is used if T_i is tangent to the (n,n)-point at p_i.

An obvious generalization is used if there are other singularities.

Lemma 4.1.2 *Let C be a smooth conic and $p_0 \notin C$, $p_1, \ldots, p_4 \in C$ be distinct points. Consider the points $p_5, p_6 \in C$ such that the lines through p_0, p_5 and p_0, p_6 are tangent to C.*

There exists a smooth curve Q of type $3(1,(1,1)_T^4, 1^2)$, through $p_0, \ldots, p_6$.

Proof: Let C_x, $x \in \mathbb{P}^1$, be a parametrization of the pencil of conics through $p_1, \ldots, p_4$. Let p_x^1, p_x^2 be the points of C_x (not distinct if C_x is singular) such that the lines through p_0, p_x^1 and p_0, p_x^2 are tangent to C_x. The correspondence

$$x \leftrightarrow \{p_x^1, p_x^2\}$$

gives a plane algebraic curve Q, parametrized by $x \in \mathbb{P}^1$, and a double cover $Q \to \mathbb{P}^1$. This cover is ramified over four points, corresponding to the three degenerate conics which contain the points $p_1, \ldots, p_4$ plus the conic which contains p_0. Therefore, by the Hurwitz formula, Q is a cubic.

The conic through $p_0, \ldots, p_4$ is not tangent to the line T_i (through p_0, p_i) at p_0, thus also Q is not tangent to T_i at p_0, $i = 1, \ldots, 4$. Since each conic C_x can be tangent to T_i only at p_i, $i = 1, \ldots, 4$, then Q intersects T_i only at p_0 and p_i, $i = 1, \ldots, 4$. This means that Q is tangent to T_i at p_i, $i = 1, \ldots, 4$, and then Q is smooth. □

Proposition 4.1.3 *In the notation of Notation* 4.1.1, *there exist pencils, without base components, of plane curves of type:*

[Br] a) $5(1,(2,2)_T^3)$;

b) $6(2,(2,2)_T^4)$;

c) $7(3,(2,2)_T^5)$;

d) $8(4,(2,2)_T^6)$.

Proof:

a) This is proved in [Br]. Notice that we are imposing 19 conditions to a linear system of dimension 20.

b) Let $\mathbb{A}(\mathbb{C})$ be an affine plane and $a, b, c, d \in \mathbb{C}\backslash\{0\}$ be numbers such that $a \neq c$ and $bc \neq \pm ad$. Consider the points of $\mathbb{A}$:

$$p_0 := (0,0),\ p_1 := (a,b),\ p_2 := (c,d),\ p_3 := (c,-d), p_4 := (a,-b)$$

and let T_i be the line through p_0 and p_i, $i = 1, \dots, 4$. Let C_1 be the conic through $p_1, \dots, p_4$ tangent to T_1, T_4 and C_2 be the conic through $p_1, \dots, p_4$ tangent to T_2, T_3.

The curves

$$2C_1 + T_2 + T_3 \quad \text{and} \quad 2C_2 + T_1 + T_4$$

generate a pencil whose general member is a curve of type $6(2, (2,2)_T^4)$.

c) Let $C \subset \mathbb{P}^2$ be a non-degenerate conic and $p_0 \notin C$, $p_1, \dots, p_5 \in C$ be distinct points such that the lines T_1, T_5, defined by p_0, p_1 and p_0, p_5, are tangent to C. From Lemma 4.1.2, there exists a curve Q of type $3(1, (1,1)_T^4, 1)$, through $p_0, \dots, p_5$, respectively.

The curves

$$2C + T_2 + T_3 + T_4 \quad \text{and} \quad 2Q + T_5$$

generate a pencil whose general member is a curve of type $7(3, (2,2)_T^5)$.

d) This is analogous to the previous case, but now the pencil is generated by

$$2C + T_2 + \cdots + T_5 \quad \text{and} \quad 2Q + T_1 + T_6.$$

□

4.2 Examples of double planes with $p_g = q = 1$

In what follows I describe how to obtain some equations of plane models of minimal Du Val double planes S (see Definition 1.1.8) of general type with $p_g = q = 1$ and $K^2 = 2, \dots, 8$.

If f is the equation of the branch curve corresponding to the plane model S' of S, then an equation for S' is $w^2 = f$, in the weighted projective space $\mathbb{P}((\deg f)/2, 1, 1, 1)$.

Table (4.1) lists the type of each branch curve that we are going to construct and the corresponding values of (K^2, g), where g denotes the genus of a general Albanese fibre of S.

We keep Notation 4.1.1.

Type of branch curve	(K^2, g)
$22(14, (5,5)^6_T)$	$(8,5),\ (8,4),\ (8,3)$
$[10+2i]\left(2i+2, (5,5)^i_T, 4^{6-i}\right)$ $i = 5,4,3$	$(6,4),\ (4,3),\ (2,2)$
$[10+2i](2i+2, (5,5)^i_T, (3,3), 4^{5-i})$ $i = 5,4,3$	$(7,5),\ (5,4),\ (3,3)$
$18(10, (5,5)^4_T, (3,3)^2)$	$(6,3)$

(4.1)

The bicanonical map of each of these Du Val double planes is non-birational (see Theorem 1.2.4). Since we are imposing 4-uple or $(3,3)$-points to Du Val's ancestors, it is worth noting that each corresponding Du Val's ancestor is of type $\mathcal{D}_i$, where i is the number of $(5,5)_T$-points.

First we obtain double planes with $(K^2, g) = (8,5)$ and $(K^2, g) = (7,5)$ such that the ramification curve is an Albanese fibre. From here constructions with $(K^2, g) = (6,4), (5,4), (4,3), (3,3), (2,2)$ will follow easily. A construction with $(K^2, g) = (6,3)$ and ramification curve strictly contained in two Albanese fibres will be also given. Finally we will get surfaces with $(K^2, g) = (8,4)$ or $(8,3)$. These three surfaces with $K^2 = 8$ were already obtained by F. Pollizi [Po4], using quotients by the action of a group (see Theorem 1.3.2).

In order to obtain $q = 1$, all the singularities, except the one at p_0, are chosen to be contained in a conic.

Recall from Section 2.2 that the branch locus B is contained in 4 fibres F_A^j of a fibration f_A (induced by the Albanese fibration). The projection $\overline{B} \subset \mathbb{P}^2$ is then contained in 4 elements of the pencil $\overline{f_A}$, image of f_A.

In Section 4.2.1, in order to get faster computations, our method is to try to figure out the configuration of the fibres F_A^j. Since each component of $F_A^j \setminus B$ is of even multiplicity, the problem of finding the support of F_A^j deals with curves of lower degree and with simpler singularities, hence with faster computations. In Section 4.2.2 the equation of $\overline{B}$ will be obtained directly.

The next sections of this chapter give a brief description of the principal steps. The detailed calculations are done in Appendix A, using the Computational Algebra System Magma.

4.2.1 $K^2 = 8, 6, 4, 2$ and $g = 5, 4, 3, 2$

To construct a Du Val's ancestor S of type $\mathcal{D}_6$ (it has $K^2 = 8$) it suffices to find a branch locus $\overline{B} \subset \mathbb{P}^2$ of type $22\left(14, (5,5)_T^6\right)$, with singularities at points $p_0, \dots, p_6$, respectively. In fact, since the lines T_i (through p_0, p_i) are in $\overline{B}$, it suffices to find a curve $\overline{B'}$ of type

$$16\left(8, (4,4)_T^6\right)$$

such that $\overline{B'} + \sum_1^6 T_i$ is reduced. If $p_1, \dots, p_6$ are contained in a conic, then $p_g(S) = q(S) = 1$.

Notice that the resolution of $\overline{B'}$ has self-intersection equal to zero. We will look to the case where $\overline{B'}$ is an element of a pencil $\overline{f_A}$ (this is case 3, iii) of Theorem 1.3.2). For this we are going to find points $p_0, \dots, p_6$ such that there is a curve D of type

$$6\left(2, (2,2)_T^2, (2,1)_T^4\right).$$

The divisor $2D + T_3 + \cdots + T_6$ is a good candidate for one of the divisors $\overline{F_A^j}$, referred above. Now to obtain the pencil $\overline{f_A}$ we could try to find another of the curves $\overline{F_A^j}$, but there is a simpler way: the procedure *LinSys* (see Appendix A.2) can be used to calculate the linear system of the curves of type $16\left(8, (4,4)_T^6\right)$, with singularities at $p_0, \dots, p_6$.

So let us look for D. Briefly, the steps are as follows. Let $\mathbb{A}$ be an affine plane, C be a smooth conic not containing the origin p_0 of $\mathbb{A}$ and $p_1, \dots, p_4$ be points in C. Denote by L the linear system of plane curves of type $6\left(2, (2,2)_T^2, (2,1)_T^2\right)$, with singularities at $p_0, \dots, p_4$, respectively. Let F be a general element of L and p_5, p_6 be general points of $\mathbb{A}$. We define a scheme *Sch* by imposing the following conditions:

· $p_5, p_6 \in C \bigcap F$;

· p_5, p_6 are double points of D (annulation of derivatives);

· the singularities of F at p_5, p_6 have one branch tangent to T_5, T_6;

· $p_5 \neq p_6$ and $p_5, p_6 \notin \{p_0, \dots, p_4\}$.

Now we compute the points of Sch with Magma, choosing one of the solutions for p_5, p_6, and we use the procedures defined in Appendix A.2 to compute $\overline{B'}$ (a curve of type $16(8, (4,4)_T^6))$, with singularities at $p_0, \dots, p_6$.

Finally we perform some verifications: that $\overline{B'}$ is reduced, the singularities are as expected, etc.

With this we find a minimal double plane with $p_g = q = 1$ and $K^2 = 8$. The divisor $2D + T_3 + \cdots + T_i$ is also a good candidate for one of the singular fibres $\overline{F_A^j}$ in the case where the branch locus is a curve of type

$$[10 + 2i]\left(2i + 2, (5,5)_T^i, 4^{6-i}\right), \quad i = 5, 4, 3.$$

In fact, using the points $p_0, \dots, p_6$ and the procedure $LinSys$, one can find branch locus of those types, obtaining then minimal double planes with $p_g = q = 1$, $K^2 = 6, 4, 2$ and $g = 4, 3, 2$, respectively.

The corresponding Magma commands are in Appendix A.4.1. There we use symmetry in order to obtain faster computations.

4.2.2 $K^2 = 7, 5, 3$ and $g = 5, 4, 3$

Here we construct a plane curve $\overline{B'}$ of type

$$15(7, (4,4)_T^5, (3,3))$$

such that $\overline{B} := \overline{B'} + \sum_1^5 T_i$ is reduced (see Notation 4.1.1). The double cover with branch locus $\overline{B}$ is a plane model of a Du Val double plane S with $K^2 = 7$ and $\chi = 1$. Notice that we are imposing a $(3,3)$-point to the branch locus of a Du Val's ancestor of type $\mathcal{D}_5$.

In this example the $(3,3)$-point p_6 is infinitely near to the $(5,5)$-point p_1 of $\overline{B}$ (*i.e.* there is a $(5,5,3,3)$-point at p_1) and the line T_1, through p_0, p_1, is tangent to the conic C defined by $p_1, \dots, p_5$. This implies $q(S) = 1$, because $\widetilde{C} - \sum_1^6 E_i$ is effective, where the curves E_i are the exceptional divisors corresponding to the blow-ups at the points p_i and $\widetilde{C}$ is the pullback of C.

To find $\overline{B'}$ we proceed as follows. In an affine plane $\mathbb{A}$, we fix a smooth conic C not containing the origin p_0 and we choose distinct points $p_1, \dots, p_5 \in C$ such that T_1 is tangent to C at p_1. We compute the linear system L of plane curves of type $15(7, (4,4)_T^5)$ and we resolve the base point of L at p_1, denoting the resulting linear system by L_1. The defining polynomial of $\overline{B'}$ is (the blow-down of) a linear combination of elements of L_1.

In order to obtain a $(3,3)$-point p_6 (infinitely near to p_1) one needs to impose conditions, the annulation of the derivatives up to order 3, to the elements of L_1. Also it is necessary to resolve this point and impose another triple (infinitely near) point. With all these conditions we define a matrix, which is denoted by Mt in Appendix A.4.2. To have a solution it is necessary that Mt has no maximal rank.

Let x, y be the coordinates of $\mathbb{A}$ and u, v be the coordinates of p_6 on $\mathbb{A}$. Define a scheme Sch by imposing the following conditions:

· the annulation of the maximal minors of Mt;

· p_6 infinitely near to p_1.

Now we compute the points of Sch with Magma, choosing one of the solutions for p_6, and we use the procedures defined in Appendix A.2 to compute $\overline{B'}$, with singularities at $p_0, \ldots, p_6$.

Finally we perform some verifications: that $\overline{B'}$ is reduced, the singularities are as expected, etc.

Notice that there is no need to verify the non-existence of other singularities: in the presence of another non-negligible singularity, the computation of the invariants of the corresponding double plane (using the formulas of [BPV, Section V. 22]) lead to a contradiction.

To find the pencil which induces the Albanese fibration, one computes the linear system of curves of type

$$16(8, (4,4)_T^5, (4,4)),$$

through $p_0, \ldots, p_6$. We will verify that the element of this pencil which contains $\overline{B'}$ is $T_1 + \overline{B'}$.

With this we find a minimal double plane with $p_g = q = 1$, $K^2 = 7$ and $g = 5$. Using again the Magma procedures referred above, one can verify that there exist also branch loci of type

$$18\left(10, (5,5)_T^4, 4, (3,3)\right) \quad \text{and} \quad 16\left(8, (5,5)_T^3, 4^2, (3,3)\right),$$

with singularities at $p_0, \ldots, p_6$. The pencils which induce the Albanese fibration are of type

$$15\left(7, (4,4)_T^4, 4, (4,4)\right) \quad \text{and} \quad 14\left(6, (4,4)_T^3, 4^2, (4,4)\right),$$

respectively. The corresponding minimal Du Val double planes are surfaces of general type with $p_g = q = 1$ and $(K^2, g) = (5,4), (3,3)$.

The Magma commands for this section are in Appendix A.4.2.

4.2.3 $K^2 = 6$ and $g = 3$

Here we construct a curve $\overline{B'}$ of type

$$14(6, (4,4)^4_T, (3,3)^2),$$

with singularities at points $p_0, \dots, p_6$, such that the $(3,3)$-points p_5, p_6 are tangent to the conic C through $p_1, \dots, p_6$. The curve $\overline{B'}$ is the union of curves D_1 and D_2 of types

$$8(4, (2,2)^4_T, (2,2)^2) \quad \text{and} \quad 6\left(2, (2,2)^4_T, (1,1)^2\right),$$

respectively. First we construct D_2 and then we use the procedure *LinSys* to obtain D_1 (a general element of $\overline{f_A}$).

Joining the lines T_i (defined by p_0, p_i) to $\overline{B'}$ one obtains a branch curve $\overline{B}$ of type $18(10, (5,5)^4_T, (3,3)^2)$. The corresponding minimal Du Val double plane has $p_g = q = 1$, $K^2 = 6$ and $g = 3$.

To find D_2, follow the steps used in Section 4.2.1, but now with L the linear system of curves of type $6\left(2, (2,2)^4_T\right)$ and *Sch* defined by

· $p_5, p_6 \in C \bigcap F$;

· F is smooth at p_5, p_6 (non-annulation of derivatives);

· F is tangent to C at p_5, p_6;

· $p_5 \neq p_6$ and $p_5, p_6 \notin \{p_0, \dots, p_4\}$.

The details can be found in Appendix A.4.3. Again we use symmetry in order to increase speed of calculations.

4.2.4 $K^2 = 8$ and $g = 4$ or 3

Now we construct a branch locus $\overline{B}$ which is the union of 6 lines T_i with curves of types $12(6, (3,3)^6_T)$ and $4(2, (1,1)^6_T)$ (recall Notation 4.1.1). This is one of Polizzi's surfaces (case 3, ii) of Theorem 1.3.2).

Given, in an affine plane $\mathbb{A}$, a generic plane cubic C not containing the origin p_0, there are 6 points $p_1, \ldots, p_6 \in C$ such that each line T_i, defined by p_0, p_i, is tangent to C. I chose an equation F of such a cubic to use in this construction.

Using the procedure *LinSys* (see Appendix A.2), one can verify the existence of a reduced quartic G of type $4\left(2,(1,1)_T^6\right)$, through $p_0, \ldots, p_6$. Let $\overline{f_A}$ be the pencil generated by the divisors $2F + T_1 + \cdots + T_6$ and $3G$. The branch locus $\overline{B}$ is the union of a general element of $\overline{f_A}$ with G and the 6 lines T_i.

The corresponding Magma computations are in Appendix A.4.4.

Finally the other surface described by Polizzi (case 3, i) of Theorem 1.3.2): the branch locus is the union of 6 lines T_i with two curves of type $8(4,(2,2)_T^6)$. From Section 4.1, there is a pencil of such curves.

Chapter 5

ϕ_2 not composed with i

In Chapters 3 and 4 we have considered minimal smooth surfaces of general type S having an involution i such that the bicanonical map ϕ_2 of S is composed with i. Now we are going to study the remaining cases, *i.e.* we suppose that ϕ_2 is not composed with i. This means that

$$h^0(W, \mathcal{O}_W(2K_W + L)) \neq 0,$$

where W is the minimal resolution of S/i and $L \equiv \frac{1}{2}B$ is the line bundle which determines the double cover $V \to W$, as in Section 2.1.

Suppose from now on that $p_g(S) = q(S) = 1$. Notice that $p_g(P) \leq p_g(S) = 1$ and $q(P) \leq q(S) = 1$.

Let P be a minimal model of W and δ, $\overline{B} \equiv 2\delta$ and the numbers r_i be as defined in Section 2.1. Recall that t denotes the number of nodes of S/i and g is the genus of a general Albanese fibre of S.

5.1 $\mathrm{Kod}(S/i) \geq 0$

Here we give a list of possibilities for the case $\mathrm{Kod}(S/i) \geq 0$. Several examples are constructed in Chapter 6.

Proposition 5.1.1 *If* $\mathrm{Kod}(P) = 0$, *only the following cases can occur:*

a) *P is an Enriques surface and*

- $\{r_i \neq 2\} = \{4\}$, $\overline{B}^2 = 0$, $t - 2 = K_S^2 \in \{2, \ldots, 7\}$, *or*
- $\{r_i \neq 2\} = \{4, 4\}$, $\overline{B}^2 = 8$, $t = K_S^2 \in \{4, \ldots, 8\}$, *or*
- $\{r_i \neq 2\} = \{6\}$, $\overline{B}^2 = 16$, $t = K_S^2 \in \{4, \ldots, 8\}$;

b) *P is a bielliptic surface and*

- $\{r_i \neq 2\} = \emptyset$, $\overline{B}^2 = 8$, $t = 0$, $K_S^2 = 4$, *or*
- $\{r_i \neq 2\} = \{4\}$, $\overline{B}^2 = 16$, $t + 6 = K_S^2 = 6$ *or* 7, *or*
- $\{r_i \neq 2\} = \{4, 4\}$, $\overline{B}^2 = 24$, $t = 0$, $K_S^2 = 8$, *or*
- $\{r_i \neq 2\} = \{6\}$, $\overline{B}^2 = 32$, $t = 0$, $K_S^2 = 8$.

Furthermore, there are examples for

- a) with $K_S^2 = 8$;
- b) with $K_S^2 = 4$, 6, 7 or 8.

Proof : It is easy to see that P cannot be a $K3$ surface: in this case we get from Proposition 2.1.4, b) that

$$K_W^2 \geq 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2,$$

which implies $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ and $K_W^2 = 0$. This contradicts the fact $\sum(r_i - 2) = 4 \neq 0$, given by Proposition 2.1.2, a).

So, from the classification of surfaces (see *e.g.* [Be] or [BPV]), $p_g(P) = q(P) = 0$ or $p_g(P) = 0$, $q(P) = 1$ (notice that $p_g(P), q(P) \leq 1$), *i.e.* P is an Enriques surface or a bielliptic surface.

a) Suppose P is an Enriques surface: Proposition 2.1.4, a) implies that $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, with equality holding only if $K_W^2 = 0$. In this case the branch locus $\overline{B}$ is smooth, *i.e.* $\sum(r_i - 2) = 0$, which contradicts Proposition 2.1.2, a). Therefore $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ or 2.

Now the only possibilities allowed by Propositions 2.1.2 and 2.1.3, a), b) are:

1) $\sum(r_i - 2) = 2$, $\overline{B}^2 = 0$, $t = K_S^2 + 2 \geq 4$;

2) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 8$ or 16, $t = K_S^2 \geq 4$.

Moreover, if a nodal curve $A_i \subset B$ is not contracted to a point, then it is mapped onto a nodal curve of the Enriques surface P. Indeed, from the adjunction formula, $K_W A_i = 0$, which means that A_i does not intersect any (-1)-curve of W.

An Enriques surface has at most 8 disjoint (-2)-curves. In case 1), the non-negligible singularities of $\overline{B}$ are a 4-uple or $(3, 3)$-point, hence $t \leq 9$. In case 2), $t = 9$ only if $\overline{B}$ has a $(3, 3)$-point, which implies that S has an elliptic curve with negative self-intersection. Since in this case $K_S^2 = 9$, this is impossible from Proposition 1.1.6, therefore $t \leq 8$.

b) Suppose P is a bielliptic surface: from Proposition 2.1.4, a), one has $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 4$, with equality holding only if $K_W^2 = 0$. In this case we get from Proposition 2.1.2, a) that

$$\sum(r_i - 2) = 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2 = 6 \neq 0,$$

which contradicts $K_W^2 = 0$. Hence $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$.

As in a), if a (-2)-curve $A_i \subset B$ is not contracted to a point, then it is mapped onto a (-2)-curve of P. But a bielliptic surface has no (-2)-curves (from Proposition 1.1.5), thus the nodal curves of B are contracted to singularities of $\overline{B}$.

Using Propositions 2.1.2 and 2.1.3, a) one obtains the following possibilities:

1) $\sum(r_i - 2) = 0$, $\overline{B}^2 = 8$, $K_S^2 = t + 4$;

2) $\sum(r_i - 2) = 2$, $\overline{B}^2 = 16$, $K_S^2 = t + 6$;

3) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 24$, $K_S^2 = t + 8$;

4) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 32$, $K_S^2 = t + 8$.

In case 1), $t = 0$, because $\overline{B}$ has only negligible singularities. In case 2), $\overline{B}$ can have a $(3,3)$-point, thus $t = 0$ or 1. In case 3), $t = 1$ only if $\overline{B}$ has a $(3,3)$-point, but then $K_S^2 = 9$ and S has an elliptic curve, which is impossible from Proposition 1.1.6. Finally, in case 4), the only non-negligible singularity of $\overline{B}$ is a point of multiplicity 6 (from Proposition 2.1.2, b)), thus $t = 0$.

The examples are constructed in Sections 6.5, 6.7, 6.8, 6.9 and 6.11. □

Proposition 5.1.2 *If* Kod$(P) = 1$, *only the following cases can occur:*

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$ *and*

- $\{r_i\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -32$, $t - 8 = K_S^2 \in \{4, \ldots, 8\}$;

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 0$ *and*

- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t - 2 = K_S^2 \in \{2, 3, 4\}$, *or*

· $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2 \in \{4, \dots, 8\}$, *or*

· $\{r_i \neq 2\} = \{4\}$, $K_P\overline{B} = 2$, $\overline{B}^2 = -4$, $t = K_S^2 \in \{4, \dots, 8\}$;

c) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$ *and*

· $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t - 2 = K_S^2 \in \{2, \dots, 6\}$, *or*

· $\{r_i\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2 \in \{4, \dots, 8\}$.

d) $\chi(\mathcal{O}_P) = 0$, $q(P) = 1$ *and*

· $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = 4$, $t = 0$, $K_S^2 = 6$, *or*

· $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = 0$, $t = 0$, $K_S^2 = 8$, *or*

· $\{r_i \neq 2\} = \{4\}$, $K_P\overline{B} = 2$, $\overline{B}^2 = 12$, $t = 0$, $K_S^2 = 8$.

Furthermore, there exist examples for

· a) with $K_S^2 = 8$;

· b) with $K_S^2 = 4$, 6 or 7;

· c) with $K_S^2 = 8$;

· d) with $K_S^2 = 6$ or 8.

Proof : Since $p_g(P), q(P) \leq 1$, we have the following cases:

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$.
From Proposition 2.1.4, b) it is immediate that $h^0(W, \mathcal{O}_W(2K_W+L)) = 1$ and $K_W^2 = 0$ (thus $\overline{B}$ is smooth). Proposition 2.1.2 gives $K_P\overline{B} = 4$ and $\overline{B}^2 = -32$. If $K_S^2 = 9$, then the number of nodal curves of B is $t = K_S^2 + 8 = 17$, from Proposition 2.1.3, a). This is impossible because Proposition 1.1.5 implies $t \leq 16$. Proposition 2.1.3, c) gives $K_S^2 \geq 4$.

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 0$.
Proposition 2.1.4, a) implies $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, with equality only if $K_S^2 = 9$ and $K_W^2 = 0$ (hence $\sum(r_i - 2) = 0$ and $W = P$). In this case Proposition 2.1.2, a) implies $K_W B' = 6$ and then $B' \neq \emptyset$. Now $p_a(B') = 1$ (see Claim 1 in the proof of Proposition 2.1.4), thus B' is an union of elliptic components. But Proposition 1.1.6 implies that a minimal surface of general type with $\chi = 1$ and $K^2 = 9$ contains no elliptic curves. Therefore $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 2$.

Since $\mathrm{Kod}(P) = 1$, $K_P\overline{B} = 0$ implies that $\overline{B}$ is contained in the elliptic fibration of P and then S has an elliptic fibration, which is impossible because S is of general type.

So $K_P\overline{B} \neq 0$. Now Propositions 2.1.2 and 2.1.3, a) give the following possibilities:

1) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t = K_S^2 + 2$;

2) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2$;

3) $\sum(r_i - 2) = 2$, $K_P\overline{B} = 2$, $\overline{B}^2 = -4$, $t = K_S^2$.

In case 1), $t > 6$ implies $\overline{B'}^2 = \overline{B}^2 + 2t > 0$, a contradiction (see Section 2.2). Similarly $t \leq 8$, in case 2). Proposition 2.1.3, c) gives $K_S^2 \geq 4$, in this case.

In case 3), the quadruple or $(3,3)$-point of $\overline{B}$ gives rise to an elliptic curve in S, thus $K_S^2 \neq 9$, from Proposition 1.1.6. Again Proposition 2.1.3, c) implies $K_S^2 \geq 4$.

c) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$.
This is analogous to the proof of b): just notice that Proposition 2.1.4, b) excludes case 3) and implies $K_W^2 = 0$ in case 2); in case 1) is no longer true that $t \leq 6$, instead use Proposition 1.1.5 to obtain $t \leq 8$ (thus $K_S^2 \leq 6$).

d) $\chi(\mathcal{O}_P) = 0$, $q(P) = 1$.
As in b), one shows that $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$ and $K_P\overline{B} \neq 0$. Propositions 2.1.2 and 2.1.3, a) give the following possibilities:

1) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 2$, $\overline{B}^2 = 4$, $t = K_S^2 - 6$;

2) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 4$, $\overline{B}^2 = 0$, $t = K_S^2 - 8$;

3) $\sum(r_i - 2) = 2$, $K_P\overline{B} = 2$, $\overline{B}^2 = 12$, $t = K_S^2 - 8$.

As in the proof of b), the existence of a quadruple or $(3,3)$-point on $\overline{B}$ implies $K_S^2 \neq 9$, in case 3).

Consider now cases 1) and 2). From Proposition 1.1.5, P has no smooth rational curves. Any singular rational curve D of $\overline{B}$ satisfies $D^2 \leq 0$, because, since $\overline{B}$ has only negligible singularities, D is contained in fibres of a fibration $\overline{f_A}$ of P (see Section 2.2). Therefore B has no (-2)-curves, *i.e.* $t = 0$.

The examples are given in Sections 6.3, 6.4, 6.6, 6.8, 6.9 and 6.10. □

Proposition 5.1.3 *If* $\mathrm{Kod}(P) = 2$, *then* $\overline{B}$ *has at most negligible singularities and only the following cases can occur:*

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$ *and*

- $K_P\overline{B} = 0$, $\overline{B}^2 = -24$, $t = 12$, $K_S^2 = 2K_P^2$, $K_P^2 = 2, 3, 4$, *or*
- $K_P\overline{B} = 2$, $\overline{B}^2 = -28$, $t - 10 + 2K_P^2 = K_S^2 \in \{2K_P^2 + 2, \ldots, 2K_P^2 + 4\}$, $K_P^2 = 1, 2$;

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$ *and*

- $K_P\overline{B} = 0$, $\overline{B}^2 = -8$, $t = 4$, $K_S^2 = 2K_P^2$, $K_P^2 = 2, 3, 4$, *or*
- $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $K_P^2 = 2$, $t + 2 = K_S^2 \in \{6, 7, 8\}$;

c) $\chi(\mathcal{O}_P) = 1$, $q(P) = 0$ *and*

- $K_P\overline{B} = 0$, $\overline{B}^2 = -8$, $t = 4$, $K_S^2 = 2K_P^2$, $K_P^2 = 1, \ldots, 4$, *or*
- $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t + 2K_P^2 - 2 = K_S^2 \in \{2K_P^2 + 2, \ldots, 2K_P^2 + 4\}$, $K_P^2 = 1, 2$, *or*
- $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $K_P^2 = 1$, $t + 2 = K_S^2 \in \{6, 7, 8\}$.

Moreover, there exist examples for

- a) with $K_S^2 = 4$, 6, 7 or 8;
- b) with $K_S^2 = 4$.

Proof :

Claim : If $K_P\overline{B} = 0$, then $\overline{B}$ is a disjoint union of nodal curves.

Proof: As P is of general type, $\overline{B}$ is an union of nodal curves. Suppose that $\overline{B}$ has a singularity. Then it contains two nodal curves D_1, D_2 such that $D_1D_2 \geq 2$, otherwise B has a (-3)-curve, contradicting $B \equiv 0 \pmod 2$. Since $K_P^2 > 0$, $K_P(D_1 + D_2) = 0$ and $(D_1 + D_2)^2 \not< 0$, the index theorem implies that $D_1 + D_2$ is homologous to zero, a contradiction. $\diamondsuit$

Since $p_g(P), q(P) \leq 1$ and $p_g(P) \geq q(P)$, we have only the following three cases:

a) $\chi(\mathcal{O}_P)=2$, $q(P)=0$.

Propositions 2.1.2, a) and 2.1.4, b) give:

$$h^0(W,\mathcal{O}_W(2K_W+L)) = K_P^2 + K_P\delta + \frac{1}{2}\sum(r_i-2) - 1,$$

$$h^0(W,\mathcal{O}_W(2K_W+L)) \leq \frac{1}{2}K_W^2 + 1 \leq \frac{1}{2}K_P^2 + 1.$$

From this we get

$$\frac{1}{2}K_P^2 + K_P\delta + \frac{1}{2}\sum(r_i-2) \leq 2, \tag{5.1}$$

with equality only if $K_W^2 = K_P^2$. Since $K_P^2 > 0$, $K_P\delta = 0$ or 1.

If $K_P\delta = 1$, then $\sum(r_i-2)=0$ and $K_P^2 = 1$ or 2.

If $K_P\delta = 0$, then $\sum(r_i-2)=0$, from the Claim above. As $K_S^2 \leq 9$, Proposition 2.1.3 implies $h^0(W,\mathcal{O}_W(2K_W+L)) \leq 3$.

Now the result follows from Propositions 2.1.2 and 2.1.3, a). Notice that Proposition 2.1.2 gives $\overline{B}^2 \geq -2(12+2K_P\delta)$. This implies $t \leq 12+2K_P\delta$, because, since $q(P)=0$ and $\overline{B}$ has only negligible singularities, every component of $\overline{B}$ has non-positive self-intersection.

b) $\chi(\mathcal{O}_P)=1$, $q(P)=1$.

Equation (5.1) is still valid here. As $p_g(P)=q(P)=1$, then $K_P^2 \geq 2$, hence

$$K_P\delta + \frac{1}{2}\sum(r_i-2) \leq 1.$$

Using the Claim above and Proposition 2.1.2, we have
$K_P\delta = 0,\ \sum(r_i-2)=0,\ K_P^2 = 2, 3$ or 4, $\overline{B}^2 = -8,\ t=4$, or
$K_P\delta = 1,\ \sum(r_i-2)=0,\ K_P^2 = 2,\ \overline{B}^2 = -12.$
Now the result follows from Proposition 2.1.3, a) (notice that $K_P^2 = 2$ implies $t \neq 7$, by Theorem 1.1.5).

c) $\chi(\mathcal{O}_P)=1$, $q(P)=0$.

Propositions 2.1.2, a), 2.1.3, c) and 2.1.4, a) imply

$$h^0(W,\mathcal{O}_W(2K_W+L)) = K_P(K_P+\delta) + \frac{1}{2}\sum(r_i-2) \leq 4 \tag{5.2}$$

and

$$h^0(W,\mathcal{O}_W(2K_W+L)) \leq 3 + \frac{1}{3}K_W^2, \tag{5.3}$$

with equality only if $K_S^2 = 9$. As $K_P^2 \geq 1$ and $K_W^2 \leq K_P^2$, this implies

$$K_P\delta \leq 2.$$

• Suppose $K_P\delta = 0$. We have $\sum(r_i - 2) = 0$, by the Claim above. Hence $h^0(W, \mathcal{O}_W(2K_W + L)) = K_P^2 \leq 4$, by (5.2). Now from Proposition 2.1.2, b) and Proposition 2.1.3, b), we have $\overline{B}^2 = (2\delta)^2 = -8$ and $t \geq 4$. Thus $t = 4$ and, using Proposition 2.1.3, a), we conclude that

$$K_S^2 = 2K_P^2, \; 1 \leq K_P^2 \leq 4.$$

• Suppose $K_P\delta = 1$. Then $h^0(W, \mathcal{O}_W(2K_W + L)) = 4$ only if $K_S^2 = 9$, from (5.2) and (5.3). If $K_P^2 = 1/2\sum(r_i - 2) = 1$, then $K_W^2 = 0$ and $K_S^2 = 9$, by (5.3). The quadruple or $(3,3)$-point of $\overline{B}$ gives rise to an elliptic curve in S, which is impossible from Proposition 1.1.6. Therefore, using (5.2), Proposition 2.1.2, b) and Proposition 2.1.3, we have

$$K_P^2 = 2, \; \sum(r_i - 2) = 0, \; \delta^2 = -3, \; t = K_S^2 - 2 \geq 4$$

or

$$K_P^2 = 1, \; \sum(r_i - 2) = 0, \; \delta^2 = -3, \; t = K_S^2 \geq 4.$$

Hence $\overline{B}^2 = -12$ and then $t \leq 6$, because, since $q(P) = 0$ and $\overline{B}$ has only negligible singularities, every component of $\overline{B}$ has non-positive self-intersection.

• Suppose $K_P\delta = 2$. Then $K_P^2 \leq 2$, from (5.2), and $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, from (5.3). The only possibility allowed by (5.2), Proposition 2.1.2, b) and Proposition 2.1.3 is:

$$K_P^2 = 1, \; \delta^2 = -4, \; t = K_S^2 - 2 \geq 4.$$

It remains to be shown that $K_S^2 \neq 9$. In this case, the curve $\overline{B}$ has at least 8 disjoint components contained in a fibration $\overline{f_A}$ of P (see Section 2.2). These are independent in $\mathrm{Pic}(P)$ from a general fibre of $\overline{f_A}$ and from K_P, so $\mathrm{Pic}(P)$ has 10 independent classes. This is a contradiction because the second Betti number of P is

$$b_2(P) = 12\chi(\mathcal{O}_P) - K_P^2 + 4q(P) - 2 = 9.$$

The examples can be found in Sections 6.4, 6.5, 6.6, 6.7, 6.8, 6.9 and 6.11. $\square$

5.2 Kod$(S/i) = -\infty$

Let S be a minimal smooth surface of general type, with $p_g = q = 1$, having an involution i such that the bicanonical map ϕ_2 of S is not composed with i. Recall that W denotes the minimal resolution of S/i and P is a minimal model of W. One has $q(P) = 0$ or 1.

Using formulas (2.1) and (2.2) of Section 2.1, one sees that if $P = \mathbb{P}^2$ then there exists a singularity in the branch locus $\overline{B}$. Blowing up, if necessary, this singular point, we can suppose that P is an Hirzebruch surface.

Suppose Kod$(P) = -\infty$. Define k, l by

$$\overline{B} \equiv: kC + \left(\frac{ek}{2} + l\right) F,$$

where F is a rational fibre of P and C is a section with lowest self-intersection $-e$.

Proposition 5.2.1 *One has*

$$h^0(W, \mathcal{O}_W(2K_W + L)) \leq 2 + q(P)$$

and

$$\begin{cases} \sum(r_i - 2) = 2l + 2k - 16 + (14 - 2k)q(P) + 2h^0(W, \mathcal{O}_W(2K_W + L)) \\ \\ \sum(r_i - 2)(r_i - 4) = 2(k - 6)(l + 6q(P) - 6) - 8h^0(W, \mathcal{O}_W(2K_W + L)). \end{cases}$$

Moreover, $K_S^2 \geq 2 + 2q(P)$ and there are examples with $q(P) = 1$, $K_S^2 = 4, 8$ and $q(P) = 0$, $K_S^2 = 8$.

Remark 5.2.2 *Since the number of possibilities for S restricted by Proposition 5.2.1 is relatively big, I will not try to study them exhaustively. As an example, I give below a list of possibilities for the case $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ (the ones marked with $*$ do exist; a^b means that a appears b times). One can try to construct these branch curves using the methods of Chapter 4 and Appendix A, but the problem here is that the calculations become too heavy for a normal computer. I do not know if it is possible to have $h^0(W, \mathcal{O}_W(2K_W + L)) = 2$ or 3.*

$q(P) = 1:$

	k	l	$\{r_i \neq 2\}$
$*$	8	2	$\{4^2\}$
	8	4	$\{6, 4^2\}$
$*$	8	6	$\{6^2, 4^2\}$

$q(P) = 0$:

	k	l	$\{r_i \neq 2\}$	
	8	$8+2i$	$\{6^i, 4^9\}$	$i \in \{0,1,2,3\}$
	10	$7+i$	$\{6^i, 4^{10}\}$	$i \in \{0,4,5\}$
	12	$10+2i$	$\{8^i, 6^5, 4^{5-i}\}$	$i \in \{0,2,3,4\}$
$*$	12	20	$\{8^5, 6^5\}$	
	14	15	$\{8^4, 6^5\}$	
	16	$18+2i$	$\{10^{2+i}, 8^5, 6^{2-i}\}$	$i \in \{0,1,2\}$

Proof of Proposition 5.2.1: If $q(P) = 1$, Proposition 2.1.3, a) gives

$$K_S^2 \geq 2h^0(W, \mathcal{O}_W(2K_W + L)) + 2,$$

hence $K_S^2 \geq 4$ and $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$ (because $K_S^2 \leq 9$).

Consider P rational. As

$$1 = p_g(S) = p_g(W) + h^0(W, \mathcal{O}_W(K_W + L)) = h^0(W, \mathcal{O}_W(K_W + L))$$

and

$$h^0(W, \mathcal{O}_W(2K_W + L)) = h^0(W, \mathcal{O}_W((K_W + L) - (-K_W))),$$

then

$$h^0(W, \mathcal{O}_W(2K_W + L)) > 1 \Rightarrow h^0(W, \mathcal{O}_W(-K_W)) = 0.$$

But the Riemann-Roch theorem gives $h^0(W, \mathcal{O}_W(-K_P)) \geq 9$, therefore the map $W \to P$ must contract at least 9 (-1)-curves, *i.e.* $K_W^2 \leq -1$. Now using Proposition 2.1.4, a) we conclude that

$$h^0(W, \mathcal{O}_W(2K_W + L)) \leq 2.$$

The rest of the proof is given by Proposition 2.1.2, using

$$K_P \equiv -2C - (e + 2 - 2q(P))F.$$

The examples are constructed in Sections 6.10, 6.11 and 6.12. □

5.3 ϕ_2 birational

In Sections 6.8 and 6.9 we present examples of surfaces which allow us to enunciate the following:

Theorem 5.3.1 *There are smooth minimal surfaces of general type with* $p_g = q = 1$, $K^2 = 6, 7$ *and birational bicanonical map.*

These surfaces were constructed to provide examples for Proposition 5.1.2, b), *i.e.* for the case $\mathrm{Kod}(W) = 1$, $p_g(W) = q(W) = 0$. The bicanonical map ϕ_2 is not composed with any of the 3 involutions associated to the bidouble cover; we verify that ϕ_2 is birational.

The surface of Section 6.6 is also an example for Proposition 5.1.2, b), but this one contains a genus 2 fibration, therefore ϕ_2 is not birational.

Chapter 6

Examples of (bi)double covers with $p_g = q = 1$

The next sections contain constructions of surfaces of general type S with $p_g = q = 1$ which are examples for Theorem 3.2.1 and Propositions 5.1.1, 5.1.2, 5.1.3 and 5.2.1. Each one is the smoth minimal model V of a bidouble cover of a ruled surface (irregular in Sections 6.10 and 6.11), except for the example in Section 6.12, which is a double plane.

Sections 6.1 and 6.2 contain examples for Theorem 3.2.1 a) (i), (ii);
the surfaces constructed in Sections 6.3, 6.4 and 6.5 are Du Val double planes which have other interesting involutions, giving examples for the propositions referred above;
Section 6.6 contains the construction of a surface with $K^2 = 4$, $g = 2$ and $\deg(\phi_2) = 2$ (thus it is not the example in [Ca2], for which ϕ_2 is composed with the three involutions associated to the bidouble cover);
in Section 6.7 a new surface with $K^2 = 8$ is obtained (it is not a standard isotrivial fibration);
Sections 6.8 and 6.9 contain the construction of new surfaces with $K^2 = 7, 6$ and $\deg(\phi_2) = 1$;
bidouble covers of irregular ruled surfaces give interesting examples in Sections 6.10 and 6.11;
finally, the equation of a double plane having bicanonical map not composed with the associated involution is obtained in Section 6.12.

Following [Ca2] or [Pa], to define a bidouble cover $V \to X$, with V, X smooth surfaces, it suffices to present:

· smooth divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;

· line bundles L_1, L_2, L_3 such that $2L_i \equiv D_j + D_k$ for each permutation (i, j, k) of $(1, 2, 3)$.

In Section 1.1 we recall the necessary formulas to compute the invariants of V.

Denote by i_1, i_2, i_3 the involutions of V corresponding to L_1, L_2, L_3, respectively. In each example the invariants of $W_j := V/i_j$, $j = 1, 2, 3$, are calculated.

We keep the notation as in Notation 4.1.1 and we use:

Notation 6.0.2 *Let $p_0, \ldots, p_j, \ldots, p_{j+s} \in \mathbb{P}^2$ be as in Notation 4.1.1 and $p'_1, \ldots, p'_j$ be the infinitely near points to $p_1, \ldots, p_j$, respectively.*

We denote by

$$\mu : X \to \mathbb{P}^2$$

the blow-up with centers

$$p_0, p_1, p'_1, \ldots, p_j, p'_j, p_{j+1}, \ldots, p_{j+s}$$

and by

$$E_0, E_1, E'_1, \ldots, E_j, E'_j, E_{j+1}, \ldots, E_{j+s}$$

the corresponding exceptional divisors (with self-intersection -1).

The notation $\tilde{\cdot}$ stands for the total transform $\mu^*(\cdot)$ of a curve.

The letter T is reserved for a general line of $\mathbb{P}^2$.

As before the letter g denotes the genus of a general Albanese fibre of S.

6.1 $K^2 = 4$, $g = 2$, W_1 ruled, W_2 rational, $\mathrm{Kod}(W_3) = 1$

Here we have the construction of a surface of general type S with $p_g = q = 1$, $K^2 = 4$ and $g = 2$ as the minimal model of a bidouble cover $V \to X$.

The quotients W_1, W_2, W_3 satisfy:

· $\mathrm{Kod}(W_1) = -\infty$, $q(W_1) = 1$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = 1$, $p_g(W_3) = 1$, $q(W_3) = 0$.

This gives an example for Theorem 3.2.1, a), (ii).

Step 1: *Construction of S.*
Let $p_0, \ldots, p_3 \in \mathbb{P}^2$ be distinct points such that p_0, p_2, p_3 are colinear and let T_i be the line through p_0, p_i, $i = 1, 2$. Denote by h the pencil of conics through p_1, p_2, p_3 which are tangent to T_1 at p_1. Let C be a general element of h and Q be the union of two general elements of h (hence C is of type $2(0, (1,1)_T, 1^2)$ and Q is of type $4(0, (2,2)_T, 2^2)$). Let T_3, T_4, T_5 be general lines through p_0.

Set

$$
\begin{aligned}
D_1 &:= \widetilde{T}_3 + \widetilde{T}_4 + \widetilde{T}_5 - 3E_0 + (E_1 - E_1'),\\
D_2 &:= \widetilde{T}_1 + \widetilde{Q} - E_0 - 3E_1 - 3E_1' - 2E_2 - 2E_3,\\
D_3 &:= \widetilde{T}_2 + \widetilde{C} - E_0 - E_1 - E_1' - 2E_2 - 2E_3.
\end{aligned}
$$

We have

$$
\begin{aligned}
L_1 &\equiv 4\widetilde{T} - E_0 - 2E_1 - 2E_1' - 2E_2 - 2E_3,\\
L_2 &\equiv 3\widetilde{T} - 2E_0 - E_1' - E_2 - E_3,\\
L_3 &\equiv 4\widetilde{T} - 2E_0 - E_1 - 2E_1' - E_2 - E_3,
\end{aligned}
$$

$$
\begin{aligned}
K_X + L_1 &\equiv \widetilde{T} - E_1 - E_1' - E_2 - E_3,\\
K_X + L_2 &\equiv -E_0 + E_1,\\
K_X + L_3 &\equiv \widetilde{T} - E_0 - E_1'
\end{aligned}
$$

and then

$$
p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 0 + 0 + 1 = 1,
$$

$$
\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1.
$$

Since

$$
\begin{aligned}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 0,\\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 0,\\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 0,
\end{aligned}
$$

the bicanonical map of V is composed with each of the involutions i_1, i_2, i_3.

Step 2: *Calculation of K_S^2.*
We have
$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$
where
$$N_1 := \widetilde{T_1} + \widetilde{T_2} - 2E_0 - 2E_1' - E_2 - E_3,$$
$$N_2 := 3\widetilde{T} - E_0 - E_1 - E_1' - E_2 - E_3.$$
As the support of the pullback of N_1 is a disjoint union of $6 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$, then
$$K_S^2 = N^2 - N_1^2 = N_2^2 = 4.$$

Step 3: *The surface W_1.*
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. As
$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2}L_1(K_X + L_1) = 2 - 2 = 0$$
and
$$p_g(W_1) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0,$$
one has $q(W_1) = 1$. The pencil of conics through p_1, p_1', p_2 and p_3 induces a rational fibration of W_1, thus $\mathrm{Kod}(W_1) = -\infty$. This pencil also lifts to the (genus 2) Albanese fibration of S.

Step 4: *The surface W_2.*
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. As
$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1$$
and
$$p_g(W_2) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0,$$
one has $q(W_2) = 0$. The pencil of lines through p_0 gives a rational fibration of W_2, thus W_2 is a rational surface.

Step 5: *The surface W_3.*
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. One has
$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2}L_3(K_X + L_3) = 2$$

and

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 1.$$

The pencil of lines through p_0 gives an elliptic fibration f of W_3. The divisor K_{W_3} is linearly equivalent to the pullback of

$$K_X + L_3 \equiv \widetilde{T_1} - E_0 - E'_1,$$

thus it is equivalent to half a fibre of f plus two (-1)-curves. Then $\mathrm{Kod(W_3)} = 1$.

6.2 $K^2 = 2$, $g = 2$, W_1 ruled, W_2 rational, $\mathrm{Kod}(W_3) = 2$

Here we obtain a surface of general type S with $p_g = q = 1$, $K^2 = 2$ and $g = 2$.

The quotients W_1, W_2, W_3 satisfy:

· $\mathrm{Kod}(W_1) = -\infty$, $q(W_1) = 1$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = 2$, $p_g(W_3) = 1$, $q(W_3) = 0$; the branch locus of the cover $V \to W_3$ is an union of twelve (-2)-curves.

This gives an example for Theorem 3.2.1, a), (i).

Step 1: *Construction of S.*
Let $p_0, \ldots, p_3 \in \mathbb{P}^2$ be points in general position and T_i be the line through p_0 and p_i, $i = 1, 2, 3$. For each $j \in \{1, 2, 3\}$ let C_j be the conic through p_1, p_2, p_3 tangent to the T_i's except for T_j. Denote by Q a general element of the linear system generated by $3C_1 + 2T_1$, $3C_2 + 2T_2$ and $3C_3 + 2T_3$. The singularities of Q are a $(3, 3)$-point at p_i, tangent to T_i, $i = 1, 2, 3$, and a double point at p_0. Let T_4 be a line through p_0 transverse to Q.

Set

$$\begin{aligned} D_1 &:= \widetilde{Q} - 2E_0 - \textstyle\sum_1^3 (3E_i + 3E'_i), \\ D_2 &:= \widetilde{T}_1 + \cdots + \widetilde{T}_4 - 4E_0 - \textstyle\sum_1^3 (E_i + E'_i), \\ D_3 &:= \textstyle\sum_1^3 (E_i - E'_i). \end{aligned}$$

We have

$$\begin{aligned}
L_1 &\equiv 2\widetilde{T} - 2E_0 - \textstyle\sum_1^3 E_i',\\
L_2 &\equiv 4\widetilde{T} - E_0 - \textstyle\sum_1^3 (E_i + 2E_i'),\\
L_3 &\equiv 6\widetilde{T} - 3E_0 - \textstyle\sum_1^3 (2E_i + 2E_i'),
\end{aligned}$$

$$\begin{aligned}
K_X + L_1 &\equiv -\widetilde{T} - E_0 + \textstyle\sum_1^3 E_i,\\
K_X + L_2 &\equiv \widetilde{T} - \textstyle\sum_1^3 E_i',\\
K_X + L_3 &\equiv 3\widetilde{T} - 2E_0 - \textstyle\sum_1^3 (E_i + E_i')
\end{aligned}$$

and then

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 0 + 0 + 1 = 1,$$

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1.$$

Since

$$\begin{aligned}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 0,\\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 0,\\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 0,
\end{aligned}$$

the bicanonical map of V is composed with each of the involutions i_1, i_2, i_3, where $V \to X$ is the bidouble cover determined by the curves D_i.

Let S be the minimal model of V.

Step 2: *Calculation of K_S^2.*

We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3} - 3E_0 - \sum_1^3 2E_i',$$

$$N_2 := 3\widetilde{T} - E_0 - \sum_1^3 (E_i + E_i').$$

As the support of the pullback of N_1 is a disjoint union of $12 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1 N_2 = 0$, one has

$$K_S^2 = N^2 - N_1^2 = N_2^2 = 2.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. It is easy to see that W_1 is a ruled surface with $q = 1$.

Step 4: *The surface* W_2.
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. As

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1$$

and

$$p_g(W_2) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0,$$

it follows $q(W_2) = 0$.

Since

$$2K_X + 2L_2 \equiv 2\widetilde{T} - \sum_1^3 2E_i'$$

and

$$2K_X + L_2 \equiv -2\widetilde{T} + E_0 + \sum_1^3 E_i,$$

one has

$$h^0(W_2, \mathcal{O}_{W_2}(2K_{W_2})) = h^0(X, \mathcal{O}_X(2K_X + 2L_2)) + h^0(X, \mathcal{O}_X(2K_X + L_2)) = 0.$$

Therefore W_2 is rational.

Step 5: *The surface* W_3.
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. One has

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2}L_3(K_X + L_3) = 2$$

and

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 1.$$

As $2(K_X + L_3)^2 = -2$, by the contraction of the three (-1)-curves contained in the pullback of $T_1 + T_2 + T_3$ we get $K_{W_3'}^2 = 1$, where W_3' is the minimal model of W_3. Therefore W_3 is of general type.

The pencil of lines through p_0 lifts to the (genus 2) Albanese fibration of S.

6.3 $K^2 = 8,\ g = 3,$ W_1 ruled, W_2 rational, $\mathrm{Kod}(W_3) = 1$

Here we are going to construct a surface of general type V with $p_g = q = 1$ and $g = 3$ such that $K_S^2 = 8$, where S is the minimal model of V. We will see that the quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

· W_1 is ruled, $q(W_1) = 1$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = p_g(W_3) = 1$, $q(W_3) = 0$

and that this gives an example for Propositions 5.1.2, a) and 5.2.1.

The surface S is one of Polizzi's Du Val double planes (see Theorem 1.3.2, 3.i)).

We use Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*
Let Q be a reduced curve of type $4(0, (2, 2)_T^2)$, *i.e.* Q is the union of two conics tangent to the lines T_1 and T_2 at p_1, p_2. Let C be another non-degenerate conic tangent to T_1, T_2 at p_1, p_2 and let $T_3, \ldots, T_6 \neq T_1, T_2$ be distinct lines through $p_0 \in T_1 \bigcap T_2$.

Set:

$$\begin{aligned} D_1 &:= \widetilde{T_1} + \cdots + \widetilde{T_6} - \textstyle\sum_1^2 (E_i + E_i') - 6E_0, \\ D_2 &:= \widetilde{Q} - \textstyle\sum_1^2 (E_i + 3E_i'), \\ D_3 &:= \widetilde{C} - \textstyle\sum_1^2 (E_i + E_i'). \end{aligned}$$

We have

$$\begin{aligned} L_1 &\equiv 3\widetilde{T} - \textstyle\sum_1^2 (E_i + 2E_i'), \\ L_2 &\equiv 4\widetilde{T} - \textstyle\sum_1^2 (E_i + E_i') - 3E_0, \\ L_3 &\equiv 5\widetilde{T} - \textstyle\sum_1^2 (E_i + 2E_i') - 3E_0 \end{aligned}$$

and

$$\begin{aligned} K_X + L_1 &\equiv -E_1' - E_2' + E_0, \\ K_X + L_2 &\equiv \widetilde{T} - 2E_0, \\ K_X + L_3 &\equiv 2\widetilde{T} - E_1' - E_2' - 2E_0. \end{aligned}$$

Let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V. We have

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L)) = 0 + 0 + 1 = 1,$$

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1.$$

One has

$$h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0,$$
$$h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 1,$$
$$h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 0,$$

thus the bicanonical map of V is composed with the involution i_2 and is not composed with the involutions i_1 and i_3.

Step 2: *Calculation of* K_S^2.
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \widetilde{T_2} - 2E_0 - \sum_1^2 2E_i',$$

$$N_2 := 4\widetilde{T} - 2E_0 - \sum_1^2 (E_i + E_i').$$

Since the support of the pullback of N_1 is a disjoint union of $8 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$,

$$K_S^2 = K_V^2 + (-N_1^2) = N^2 - N_1^2 = N_2^2 = 8$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. The pencil of conics tangent to T_1, T_2 at p_1, p_2 lifts to a rational fibration of W_1 (and to the Albanese fibration of S, which is of genus 3). One has

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2}L_1(K_X + L_1) = 2 - 2 = 0$$

and then W_1 is a ruled surface with $q = 1$.

Step 4: *The surface* W_2.
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. One has that W_2 is ruled and

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1,$$

hence W_2 is a rational surface.

Step 5: *The surface* W_3.
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. The canonical divisor K_{W_3} is the pullback of $2\widetilde{T} - E_1' - E_2' - 2E_0$ and

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2} L_3(K_X + L_3) = 2,$$

thus $\operatorname{Kod}(W_3) = p_g(W_3) = 1$ and $q(W_3) = 0$.

6.4 $K^2 = 6$, $g = 4$, $\operatorname{Kod}(W_1) = 2$, W_2 rational, $\operatorname{Kod}(W_3) = 1$

This section contains the construction of a bidouble cover $V \to X$ such that the minimal model S of V is a surface of general type with $K^2 = 6$, $p_g = q = 1$, $g = 4$ and that the quotients $W_j := V/i_j$ satisfy:

· $\operatorname{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;

· W_2 is rational;

· $\operatorname{Kod}(W_3) = 1$, $p_g(W_3) = 0$, $q(W_3) = 1$.

This is an example for Propositions 5.1.2, d) and 5.1.3, a).

One can verify that S is the Du Val double plane obtained imposing a 4-uple point to the branch locus of a Du Val's ancestor of type $\mathcal{D}_5$ (*cf.* Section 4.2.1).

Recall Notations 4.1.1 and 6.0.2.

Step 1: *Construction of* S.
From Proposition 4.1.3 in Section 4.1, there is a pencil l, with no base component, of curves of type $7(3, (2, 2)_T^5)$. Let Q be a general element of this pencil and C be a reduced curve of type $4(2, (1, 1)_T^5)$.

Set:

$$\begin{aligned}
D_1 &:= \widetilde{\widetilde{T_1}} + \cdots + \widetilde{\widetilde{T_4}} - \textstyle\sum_1^4 (E_i + E_i') + (E_5 - E_5') - 4E_0,\\
D_2 &:= \widetilde{\widetilde{T_5}} + \widetilde{Q} - \textstyle\sum_1^4 (E_i + 3E_i') - 3E_5 - 3E_5' - 4E_0,\\
D_3 &:= \widetilde{C} - \textstyle\sum_1^5 (E_i + E_i') - 2E_0.
\end{aligned}$$

We have

$$\begin{aligned}
L_1 &\equiv 6\widetilde{T} - \textstyle\sum_1^4 (E_i + 2E_i') - 2E_5 - 2E_5' - 3E_0,\\
L_2 &\equiv 4\widetilde{T} - \textstyle\sum_1^4 (E_i + E_i') - E_5' - 3E_0,\\
L_3 &\equiv 6\widetilde{T} - \textstyle\sum_1^5 (E_i + 2E_i') - 4E_0
\end{aligned}$$

and

$$\begin{aligned}
K_X + L_1 &\equiv \left(2\widetilde{T} - \textstyle\sum_1^4 E_i' - E_0\right) + \left(\widetilde{T_5} - E_5 - E_5' - E_0\right),\\
K_X + L_2 &\equiv \widetilde{T} + E_5 - 2E_0,\\
K_X + L_3 &\equiv 3\widetilde{T} - \textstyle\sum_1^5 E_i' - 3E_0,
\end{aligned}$$

hence

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1,$$

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 + 0 - 1 - 2 = 1.$$

One has

$$\begin{aligned}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 0,\\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 2,\\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 0,
\end{aligned}$$

thus the bicanonical map of V is composed with the involution i_2 and is not composed with the involutions i_1 and i_3.

Step 2: *Calculation of K_S^2.*
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \cdots + \widetilde{T_5} - 5E_0 - \sum_1^5 2E_i',$$

$$N_2 := 5\widetilde{T} - 3E_0 - \sum_1^5 (E_i + E_i').$$

As the support of the pullback of N_1 is a disjoint union of $20 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$, then

$$K_S^2 = K_V^2 + (-N_1^2) = N^2 - N_1^2 = N_2^2 = 6.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. One has

$$p_g(W_1) = h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2} L_1(K_X + L_1) = 2.$$

Since $2(K_X + L_1)^2 = -2$, we get, by contraction of the five (-1)-curves contained in the pullback of $T_1 + \cdots + T_5$, $K^2_{W_1'} = 3$, where W_1' is the minimal model of W_1. Therefore W_1 is of general type.

Step 4: *The surface* W_2.
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. The pencil of lines through p_0 lifts to a rational pencil of W_2. As

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2} L_2(K_X + L_2) = 2 - 1 = 1,$$

then W_2 is rational.

Step 5: *The surface* W_3.
Let W_3 be the double cover of X with branch locus $D_1 + D_2$ and W_3' be a minimal model of W_3. The equivalence

$$2K_X + 2L_3 \equiv \sum_1^5 \left(\widetilde{T}_i - 2E_i' - E_0\right) + \left(\widetilde{T} - E_0\right)$$

implies that $2K_{W_3'}$ is linearly equivalent to a fibre of an elliptic fibration of W_3, hence $\mathrm{Kod}(W_3) = 1$. Also

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 0,$$

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2} L_3(K_X + L_3) = 2 - 2 = 0$$

and then $q(W_3) = 1$. The pencil l lifts to a genus 2 fibration of W_3 and to the (genus 4) Albanese fibration of S.

6.5 $K^2 = 4,\ g = 3,$ $\mathrm{Kod}(W_1) = 2,\ W_2$ rational, $\mathrm{Kod}(W_3) = 0$

In this section a surface of general type S with $p_g = q = 1$, $K^2 = 4$ and $g = 3$ is constructed. It is the minimal model of a double cover of surfaces W_1, W_2, W_3 such that:

· $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 1$.

This gives an example for Propositions 5.1.1, b) and 5.1.3, a).

One can verify that S is the Du Val double plane obtained imposing two 4-uple points to the branch locus of a Du Val's ancestor of type $\mathcal{D}_4$ (*cf.* Section 4.2.1).

We keep Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*
From Proposition 4.1.3, there is a pencil l, with no base component, of curves of type $6(2,(2,2)_T^4)$, through points $p_0, \ldots, p_4$ (*i.e.* of plane curves of degree 6 with a double point at p_0 and a tacnode at p_i tangent to the line through p_0, p_i, $i = 1, \ldots, 4$). Let Q be a general element of this pencil, C be a reduced curve of type $4(2,(1,1)_T^4)$ and set:

$$\begin{aligned} D_1 &:= \widetilde{T_1} + \cdots + \widetilde{T_4} - \textstyle\sum_1^4 (E_i + E_i') - 4E_0, \\ D_2 &:= \widetilde{Q} - \textstyle\sum_1^4 (E_i + 3E_i') - 2E_0, \\ D_3 &:= \widetilde{C} - \textstyle\sum_1^4 (E_i + E_i') - 2E_0. \end{aligned}$$

Let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V.

We have

$$\begin{aligned} L_1 &\equiv 5\widetilde{T} - \textstyle\sum_1^4 (E_i + 2E_i') - 2E_0, \\ L_2 &\equiv 4\widetilde{T} - \textstyle\sum_1^4 (E_i + E_i') - 3E_0, \\ L_3 &\equiv 5\widetilde{T} - \textstyle\sum_1^4 (E_i + 2E_i') - 3E_0 \end{aligned}$$

and

$$\begin{aligned} K_X + L_1 &\equiv 2\widetilde{T} - \textstyle\sum_1^4 E_i' - E_0, \\ K_X + L_2 &\equiv \widetilde{T} - 2E_0, \\ K_X + L_3 &\equiv 2\widetilde{T} - \textstyle\sum_1^4 E_i' - 2E_0. \end{aligned}$$

Then

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1,$$

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2} \sum_1^3 L_i(K_X + L_i) = 4 + 0 - 1 - 2 = 1.$$

As

$$\begin{aligned} h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 0, \\ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 1, \\ h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 0, \end{aligned}$$

the bicanonical map of V is composed with the involution i_2, corresponding to L_2, and is not composed with the involutions i_1, i_3, corresponding to L_1, L_3.

Step 2: *Calculation of* K_S^2.
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - 4E_0 - \sum_1^4 2E_i',$$

$$N_2 := 4\widetilde{T} - 2E_0 - \sum_1^4 (E_i + E_i').$$

As the support of the pullback of N_1 is a disjoint union of $16 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$,

$$K_S^2 = K_V^2 + (-N_1^2) = N^2 - N_1^2 = N_2^2 = 4.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. We have

$$p_g(W_1) = h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2} L_1(K_X + L_1) = 2 + 0 = 2.$$

Since $2(K_X + L_1)^2 = -2$, we get, by contraction of the eight (-1)-curves contained in the pullback of $T_1 + \cdots + T_4$, $K_{W_1'}^2 = 6$, where W_1' is the minimal model of W_1. Therefore W_1 is of general type.

Step 4: *The surface* W_2.

Let W_2 be the double cover of X with branch locus $D_1 + D_3$. The pencil of lines through p_0 lifts to a rational fibration of W_2. As

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1,$$

W_2 is rational.

Step 5: *The surface* W_3.
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. The divisor $2K_{W_3}$ is linearly equivalent to the pullback of the divisor

$$\widetilde{T_1} + \cdots + \widetilde{T_4} - \sum_1^4 2E_i' - 4E_0,$$

whose support is a disjoint sum of (-1)-curves. Hence $\mathrm{Kod}(W_3) = 0$. One has

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 0$$

and

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2}L_3(K_X + L_3) = 2 - 2 = 0.$$

The pencil l lifts to an elliptic fibration of W_3 and to the (genus 3) Albanese fibration of S.

6.6 $K^2 = 4$, $g = 2$, W_1 ruled, $\mathrm{Kod}(W_2) = 1$, $\mathrm{Kod}(W_3) = 2$

This section contains the construction of a surface of general type S with $p_g = q = 1$, $K^2 = 4$, $g = 2$ and $\deg(\phi_2) = 2$. We will verify that S is the minimal model of a double cover of surfaces W_1, W_2, W_3 such that:

· W_1 is ruled, $q(W_1) = 1$;

· $\mathrm{Kod}(W_2) = 1$, $p_g(W_2) = q(W_2) = 0$;

· $\mathrm{Kod}(W_3) = 2$, $p_g(W_3) = 1$, $q(W_3) = 0$.

This gives an example for Propositions 5.1.2, b) and 5.1.3, a).
Notations 4.1.1 and 6.0.2 are used.

Step 1: *Construction of S.*

By Proposition 4.1.3 in Section 4.1, there is a pencil l, with no base component, of curves of type $6(2,(2,2)^4_T)$. Let Q_1 be a general element of this pencil, Q_2 be a curve of type $3(1,(1,1)^4_T)$ and $Q := Q_1 + Q_2$. Let T_5 be a line through p_0 transverse to Q.

Set:

$$\begin{aligned} D_1 &:= \widetilde{T_1} + \widetilde{Q} - 4E_1 - 4E_1' - \textstyle\sum_2^4(3E_i + 3E_i') - 4E_0,\\ D_2 &:= \widetilde{T_2} + \cdots + \widetilde{T_5} - \textstyle\sum_2^4(E_i + E_i') - 4E_0,\\ D_3 &:= \textstyle\sum_2^4(E_i - E_i'). \end{aligned}$$

We have

$$\begin{aligned} L_1 &\equiv 2\widetilde{T} - \textstyle\sum_2^4 E_i' - 2E_0,\\ L_2 &\equiv 5\widetilde{T} - 2E_1 - 2E_1' - \textstyle\sum_2^4(E_i + 2E_i') - 2E_0,\\ L_3 &\equiv 7\widetilde{T} - \textstyle\sum_1^4(2E_i + 2E_i') - 4E_0 \end{aligned}$$

and

$$\begin{aligned} K_X + L_1 &\equiv -\widetilde{T} + \textstyle\sum_1^4 E_i + E_1' - E_0,\\ K_X + L_2 &\equiv 2\widetilde{T} - \textstyle\sum_1^4 E_i' - E_1 - E_0,\\ K_X + L_3 &\equiv \widetilde{T_1} + \cdots + \widetilde{T_4} - \textstyle\sum_1^4(E_i + E_i') - 3E_0. \end{aligned}$$

Let $\psi : V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V. Then:

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 0 + 0 + 1 = 1,$$

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 - 2 - 1 + 0 = 1.$$

One can verify that

$$\begin{aligned} h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 0,\\ h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 0,\\ h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 1, \end{aligned}$$

thus the bicanonical map of V is composed with the involution i_1 and is not composed with the involutions i_2 and i_3.

Step 2: *Calculation of K_S^2.*
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - 4E_0 - (E_1 + E_1') - \sum_2^4 2E_i'$$

and

$$N_2 := 4\widetilde{T} - 2E_0 - \sum_1^4 (E_i + E_i').$$

As the support of the pullback of N_1 is a disjoint union of $14 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1 N_2 = 0$, then

$$K_S^2 = N^2 + (-N_1^2) = N_2^2 = 4.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. The pencil of lines through p_0 gives a rational fibration of W_1. Since

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2} L_1(K_X + L_1) = 2 - 2 = 0,$$

then $q(W_1) = 1$.

The above pencil lifts to the (genus 2) Albanese fibration of S.

Step 4: *The surface* W_2.
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. We have

$$p_g(W_2) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0$$

and

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2} L_2(K_X + L_2) = 2 - 1 = 1.$$

Since $2(K_X + L_2)^2 = -4$, we get, by contraction of the four (-1)-curves contained in the pullback of $T_1 + \cdots + T_4$, a surface W_2' such that $K_{W_2'}^2 = 0$. Notice that W_2 is not of general type, because

$$h^0(X, \mathcal{O}_X(2K_X + 2L_2)) = h^0(X, \mathcal{O}_X(2K_X + L_2)) = 0$$

implies $h^0(W_2, \mathcal{O}_{W_2}(2K_{W_2})) = 0$. Therefore W_2' is a minimal model of W_2. The pencil of lines through p_0 lifts to a genus 2 pencil of W_2', hence $\mathrm{Kod}(W_2) > 0$ and then $\mathrm{Kod}(W_2) = 1$.

Step 5: *The surface W_3.*
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. We have

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 1$$

and

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2} L_3(K_X + L_3) = 2 + 0 = 2.$$

Since $2(K_X + L_3)^2 = -2$, we get, by contraction of the four (-1)-curves contained in the pullback of $T_1 + \cdots + T_4$, $K^2_{W_3'} = 2$, where W_3' is the minimal model of W_3.Therefore W_3 is of general type.

Step 6: *Degree of ϕ_2.*
The system $|\psi^*(N)|$ is strictly contained in the bicanonical system of V. Since ϕ_2 is composed with i_1 and the map $\tau : X \dashrightarrow \mathbb{P}^2$ induced by $|N|$ is birational (this can be verified using the Magma function *IsInvertible*), one has $\deg(\phi_2) = 2$.

6.7 $K^2 = 8,\ g = 3,$ $\mathrm{Kod}(W_1) = 2,\ \mathrm{Kod}(W_2) = 0,\ \mathrm{Kod}(W_3) = 0$

A smooth projective surface S of general type is said to be a *standard isotrivial fibration* if there exists a finite group G which acts faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. The paper [Po1] contains examples of such surfaces with $K^2 = 8$.

This section contains the construction of the first surface of general type S with $p_g = q = 1$, $K^2 = 8$ and $g = 3$ which is not a standard isotrivial fibration.

The surface S is the minimal model of a double cover of surfaces W_1, W_2, W_3 such that:

· $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;

· $\mathrm{Kod}(W_2) = 0$, $p_g(W_2) = 0$, $q(W_2) = 1$;

· $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 0$.

This is an example for Propositions 5.1.1 a), b) and 5.1.3 a).

Recall Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*

Let G be a curve of type $6(2,(2,2)_T^4)$ and C be a curve of type $8(4,(2,2)_T^4,(3,3))$ such that $G+C$ is reduced and the $(3,3)$-point of C is tangent to G. The existence of these curves is shown in Appendix A.3.

Set:

$$\begin{aligned}
&D_1 := \widetilde{T_1} + \widetilde{T_2} - \textstyle\sum_1^2 2E_i' + (E_5 - E_5') - 2E_0,\\
&D_2 := \widetilde{G} - \textstyle\sum_1^4 (2E_i + 2E_i') - (E_5 + E_5') - 2E_0,\\
&D_3 := \widetilde{T_3} + \widetilde{T_4} + \widetilde{C} - \textstyle\sum_1^2 (2E_i + 2E_i') - \sum_3^4 (2E_i + 4E_i') - (3E_5 + 3E_5') - 6E_0.
\end{aligned}$$

We have

$$\begin{aligned}
&L_1 \equiv 8\widetilde{T} - \textstyle\sum_1^2 (2E_i + 2E_i') - \sum_3^4 (2E_4 + 3E_4') - (2E_5 + 2E_5') - 4E_0,\\
&L_2 \equiv 6\widetilde{T} - \textstyle\sum_1^5 (E_i + 2E_i') - 4E_0,\\
&L_3 \equiv 4\widetilde{T} - \textstyle\sum_1^2 (E_i + 2E_i') - \sum_3^4 (E_i + E_i') - E_5' - 2E_0
\end{aligned}$$

and

$$\begin{aligned}
&K_X + L_1 \equiv 5\widetilde{T} - \textstyle\sum_1^2 (E_i + E_i') - \sum_3^4 (E_4 + 2E_4') - (E_5 + E_5') - 3E_0,\\
&K_X + L_2 \equiv 3\widetilde{T} - \textstyle\sum_1^5 E_i' - 3E_0,\\
&K_X + L_3 \equiv \widetilde{T} - \textstyle\sum_1^2 E_i' + E_5 - E_0.
\end{aligned}$$

Let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V. Then

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 + 0 - 2 - 1 = 1.$$

The procedure *LinSys* of Appendix A.2 can be used to confirm that

$$h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$\begin{aligned}
&h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0,\\
&h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 1,\\
&h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 2,
\end{aligned}$$

therefore

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1$$

and the bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3.

The Albanese fibration of S is induced by a pencil of curves of type $14(6,(4,4)_T^4,(4,4))$, which contains a fibre equal to $G + C$ (see Appendix A.3).

From [Po2, Theorem 3.2], the existence of such reducible fibre implies that S is not a standard isotrivial fibration, so this is not one of Polizzi's examples.

Step 2: *Calculation of K_S^2.*
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - 4E_0 - \sum_1^4 2E_i' + (E_5 - E_5')$$

and

$$N_2 := 8\widetilde{T} - 4E_0 - \sum_1^5 (2E_i + 2E_i').$$

As the support of the pullback of N_1 is a disjoint union of $18 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component (again this can be confirmed with the Magma procedure *LinSys*) and $N_1 N_2 = 0$,

$$K_S^2 = N^2 + (-N_1^2) = N_2^2 = 8.$$

Step 3: *The surface W_1.*
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. We have

$$p_g(W_1) = h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2} L_1(K_X + L_1) = 2 + 0 = 2.$$

Since $2(K_X + L_1)^2 = 0$, we get, by contraction of the four (-1)-curves contained in the pullback of $\widetilde{T_3} + \widetilde{T_4}$, $K_{W_1'}^2 = 4$, where W_1' is the minimal model of W_1. Therefore W_1 is of general type.

Step 4: *The surface W_2.*
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. One has

$$p_g(W_2) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0$$

and

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2} L_2(K_X + L_2) = 2 - 2 = 0.$$

Let T_5 be the line through p_0, p_5. From

$$2(K_X + L_2) \equiv \left(\widetilde{T_1} + \cdots + \widetilde{T_4} - 4E_0 - \sum_1^4 2E_i'\right) + \left(2\widetilde{T_5} - 2E_0 - 2E_5'\right),$$

one sees that $2K_{W_2'} \equiv 0$, where W_2' is the minimal model of W_2. So $\mathrm{Kod}(W_2) = 0$ and then W_2 is a bielliptic surface.

Step 5: *The surface* W_3.
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. The divisor $2K_{W_3}$ is linearly equivalent to the pullback of the divisor

$$\widetilde{T_1} + \widetilde{T_2} - \sum_1^2 2E_i' - 2E_0 + 2E_5,$$

whose support is the sum of the pullback of E_5 with a disjoint sum of (-1)-curves. Hence $2K_{W_3'} \equiv 0$, where W_3' is the minimal model of W_3, and then $\mathrm{Kod}(W_3) = 0$. One has

$$p_g(W_3) = h^0(X, \mathcal{O}_X(K_X + L_3)) = 0$$

and

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2} L_3(K_X + L_3) = 2 - 1 = 1,$$

thus W_3 is an Enriques surface.

6.8 $K^2 = 7$, $g = 3$, $\mathrm{Kod}(W_1) = 2$, $\mathrm{Kod}(W_2) = 1$, $\mathrm{Kod}(W_3) = 0$

This section contains the construction of a a bidouble cover $V \to X$, with X rational, such that the minimal model S of V is a surface of general type with $K^2 = 7$, $p_g = q = 1$, $g = 3$ and birational bicanonical map.

Let i_j, $j = 1, 2, 3$, be the involutions associated to the bidouble cover. The quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

- $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;
- $\mathrm{Kod}(W_2) = 1$, $p_g(W_2) = 0$, $q(W_2) = 0$;
- $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 1$.

This is an example for Propositions 5.1.1, b), 5.1.2, b) and 5.1.3, a).

We keep Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*
From Appendix A.3, there exist a curve C of type $7(3,(2,2)_T^4,3)$ (*i.e.* C is a plane curve of degree 7 with triple points at p_0, p_5 and a tacnode at p_i tangent to the line T_i through p_0, p_i, $i = 1, \dots, 4$) and a curve G of type $6(2,(2,2)_T^4,1)$, both through points $p_0, \dots, p_5$, such that $C + G$ is reduced.

Set:

$$
\begin{aligned}
D_1 &:= \widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3} - \textstyle\sum_1^3 2E_i' + E_5 - 3E_0,\\
D_2 &:= \widetilde{T_4} + \widetilde{G} - \textstyle\sum_1^3 (2E_i + 2E_i') - (2E_4 + 4E_4') - E_5 - 3E_0,\\
D_3 &:= \widetilde{C} - \textstyle\sum_1^4 (2E_i + 2E_i') - 3E_5 - 3E_0.
\end{aligned}
$$

We have

$$
\begin{aligned}
L_1 &\equiv 7\widetilde{T} - \textstyle\sum_1^3 (2E_i + 2E_i') - (2E_4 + 3E_4') - 2E_5 - 3E_0,\\
L_2 &\equiv 5\widetilde{T} - \textstyle\sum_1^3 (E_i + 2E_i') - (E_4 + E_4') - E_5 - 3E_0,\\
L_3 &\equiv 5\widetilde{T} - \textstyle\sum_1^4 (E_i + 2E_i') - 3E_0
\end{aligned}
$$

and

$$
\begin{aligned}
K_X + L_1 &\equiv 4\widetilde{T} - \textstyle\sum_1^3 (E_i + E_i') - (E_4 + 2E_4') - E_5 - 2E_0,\\
K_X + L_2 &\equiv 2\widetilde{T} - \textstyle\sum_1^3 E_i' - 2E_0,\\
K_X + L_3 &\equiv 2\widetilde{T} - \textstyle\sum_1^4 E_i' + E_5 - 2E_0.
\end{aligned}
$$

Let $\psi : V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V. Then

$$\chi(\mathcal{O}_S) = 4 + \frac{1}{2}\sum_1^3 L_i(K_X + L_i) = 4 + 0 - 1 - 2 = 1.$$

The procedure *LinSys* of Appendix A.2 can be used to verify that

$$h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$
\begin{aligned}
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) &= 1,\\
h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) &= 1,\\
h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) &= 1,
\end{aligned}
$$

therefore

$$p_g(S) = \sum_1^3 h^0(X, \mathcal{O}_X(K_X + L_i)) = 1 + 0 + 0 = 1$$

and the bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3.

Step 2: *Calculation of* K_S^2.
We have

$$N := 2K_X + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - 4E_0 - \sum_1^4 2E_i'$$

and

$$N_2 := 7\widetilde{T} - 3E_0 - \sum_1^4 (2E_i + 2E_i') - E_5.$$

As the support of the pullback of N_1 is a disjoint union of $16 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component (use the procedure *LinSys*) and $N_1N_2 = 0$,

$$K_S^2 = N^2 + (-N_1^2) = N_2^2 = 7.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of X with branch locus $D_2 + D_3$. We have

$$p_g(W_1) = h^0(X, \mathcal{O}_X(K_X + L_1)) = 1$$

and

$$\chi(\mathcal{O}_{W_1}) = 2 + \frac{1}{2}L_1(K_X + L_1) = 2 + 0 = 2.$$

Since $2(K_X + L_1)^2 = 0$, we get, by contraction of the two (-1)-curves contained in the pullback of $\widetilde{T_4}$, $K_{W_1'}^2 = 2$, where W_1' is the minimal model of W_1. Therefore W_1 is of general type.

Step 4: *The surface* W_2.
Let W_2 be the double cover of X with branch locus $D_1 + D_3$. The pencil of lines through p_0 lifts to an elliptic fibration f of W_2. The divisor $2K_{W_2}$ is linearly equivalent to the pullback of the divisor

$$\left(\widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3} - \sum_1^3 2E_i' - 3E_0\right) + \left(\widetilde{T} - E_0\right),$$

whose support is a disjoint sum of six (-1)-curves with a fibre of f. Hence $\mathrm{Kod}(W_2) = 1$.

One has

$$p_g(W_2) = h^0(X, \mathcal{O}_X(K_X + L_2)) = 0$$

and

$$\chi(\mathcal{O}_{W_2}) = 2 + \frac{1}{2}L_2(K_X + L_2) = 2 - 1 = 1.$$

The pencil of curves of type $6(2, (2,2)_T^4)$ lifts to the (genus 3) Albanese fibration of S.

Step 5: *The surface W_3.*
Let W_3 be the double cover of X with branch locus $D_1 + D_2$. The support of the pullback of

$$2(K_X + L_3) \equiv \left(\widetilde{T_1} + \cdots + \widetilde{T_4} - \sum_1^4 2E_i' - 4E_0\right) + 2E_5$$

is a disjoint sum of (-1)-curves, hence $\mathrm{Kod}(W_3) = 0$.

One has

$$p_g(W_3) = h^0(\mathcal{O}_X(K_X + L_3)) = 0$$

and

$$\chi(\mathcal{O}_{W_3}) = 2 + \frac{1}{2}L_3(K_X + L_3) = 2 - 2 = 0.$$

Step 6: *Verification that ϕ_2 is birational*
The system $|\psi^*(N)|$ is strictly contained in the bicanonical system of V. The bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, hence it is birational if the map τ given by $|N| = N_1 + |N_2|$ is birational. This is in fact the case, see Appendix A.3, where Magma is used to show that the image of τ is of degree $7 = N_2^2$.

6.9 $K^2 = 6,\ g = 3,$ $\mathrm{Kod}(W_1) = 2,\ \mathrm{Kod}(W_2) = 1,\ \mathrm{Kod}(W_3) = 0$

One can obtain a construction analogous to the one of Example 6.8, but with $K_S^2 = 6$ instead: replace the triple point of C by a $(2,2)$-point, tangent to G. Such a curve exists, see Appendix A.3. With this change the branch locus in W_3 has a 4-uple point instead of a $(3,3)$-point.

6.10 $K^2 = 8,\ g = 3,$ $\mathrm{Kod}(W_1) = 1,\ W_2$ **ruled,** $\mathrm{Kod}(W_3) = 1$

In this section we give the construction of a surface of general type S, with $K^2 = 8$, $p_g = q = 1$ and $g = 3$, as a bidouble cover of a ruled surface Z with $q(Z) = 1$.

Let i_j, $j = 1, 2, 3$, be the involutions associated to the bidouble cover. The quotients $W_j := S/i_j$, $j = 1, 2, 3$, satisfy:

· $\mathrm{Kod}(W_1) = 1$, $p_g(W_1) = 0$, $q(W_1) = 1$;

· W_2 is ruled, $q(W_2) = 1$;

· $\mathrm{Kod}(W_3) = p_g(W_3) = q(W_3) = 1$.

This is an example for Propositions 5.1.2 c), d) and 5.2.1.

We keep Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*
Let $F_1, \ldots, F_4$ be distinct fibres of the Hirzebruch surface $\mathbb{F}_0$ and $Z \to \mathbb{F}_0$ be the double cover with branch locus $F_1 + \cdots + F_4$. Clearly Z is a ruled surface with irregularity 1. Denote by γ the rational fibration of Z.

Let $G, G_1, \ldots, G_6$ be distinct smooth elliptic sections of γ and $\Gamma_1, \ldots, \Gamma_4$ be distinct fibres of γ such that $\Gamma_1 + \Gamma_2 \equiv 2\Gamma_3 \equiv 2\Gamma_4$.

Set

$$\begin{aligned} D_1 &:= \Gamma_1 + \Gamma_2, \\ D_2 &:= G_1 + \cdots + G_4, \\ D_3 &:= G_5 + G_6 \end{aligned}$$

and

$$\begin{aligned} L_1 &:= 3G + \Gamma_3 - \Gamma_4, \\ L_2 &:= G + \Gamma_4, \\ L_3 &:= 2G + \Gamma_3. \end{aligned}$$

We have

$$\begin{aligned} K_Z + L_1 &\equiv G + \Gamma_3 - \Gamma_4, \\ K_Z + L_2 &\equiv -G + \Gamma_4, \\ K_Z + L_3 &\equiv \Gamma_3 \end{aligned}$$

and then

$$p_g(S) = \sum_1^3 h^0(Z, \mathcal{O}_Z(K_Z + L_i)) = 0 + 0 + 1 = 1,$$

$$\chi(\mathcal{O}_S) = 4 \cdot 0 + \frac{1}{2}\sum_1^3 L_i(K_Z + L_i) = 0 + 0 + 0 + 1 = 1.$$

The linear system

$$|N| := |2K_Z + \sum_1^3 L_i| \equiv |2G + 2\Gamma_3|$$

has no base component, thus

$$K_S^2 = N^2 = 8.$$

As

$$h^0(X, \mathcal{O}_X(2K_X + L_1 + L_2)) = 0,$$
$$h^0(X, \mathcal{O}_X(2K_X + L_1 + L_3)) = 2,$$
$$h^0(X, \mathcal{O}_X(2K_X + L_2 + L_3)) = 1,$$

the bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3.

Step 2: *The surface* W_1.
Let W_1 be the double cover of Z with branch locus $D_2 + D_3$, determined by L_1. We have

$$p_g(W_1) = h^0(Z, \mathcal{O}_Z(K_Z + L_1)) = 0$$

and

$$\chi(\mathcal{O}_{W_1}) = \frac{1}{2}L_1(K_Z + L_1) = 0,$$

hence $q(W_1) = 1$. Since

$$2(K_Z + L_1) \equiv 2G,$$

we have $\mathrm{Kod}(W_1) = 1$.

Step 3: *The surface* W_2.
Let W_2 be the double cover of Z with branch locus $D_1 + D_3$, determined by L_2. The ruling of Z lifts to a ruling of W_2 and to the (genus 3) Albanese fibration of S. As

$$\chi(\mathcal{O}_{W_2}) = \frac{1}{2}L_2(K_Z + L_2) = 0$$

then W_2 is ruled and $q(W_2) = 1$.

Step 4: *The surface* W_3.
Let W_3 be the double cover of Z with branch locus $D_1 + D_2$, determined by L_3. One has

$$p_g(W_3) = h^0(Z, \mathcal{O}_Z(K_Z + L_3)) = 1$$

and
$$\chi(\mathcal{O}_{W_3})=\frac{1}{2}L_3(K_Z+L_3)=1.$$
Since
$$K_Z+L_3\equiv\Gamma_3,$$
then $\mathrm{Kod}(W_1)=1$.

6.11 $K^2=4,\ g=3,$ W_1 ruled, $\mathrm{Kod}(W_2)=1,\ \mathrm{Kod}(W_3)=2$

This section contains the construction of a bidouble cover $V\to Z$, with Z ruled and $q(Z)=1$, such that the minimal model S of V is a surface of general type with $K^2=4$, $p_g=q=1$, $g=3$ and that the bicanonical map ϕ_2 of S is not composed with any of the involutions i_1,i_2,i_3 associated to the bidouble cover.

The quotients $W_j:=V/i_j$, $j=1,2,3$, satisfy:

- $\mathrm{Kod}(W_1)=-\infty$, $q(W_1)=1$;
- $\mathrm{Kod}(W_2)=0$, $p_g(W_2)=0$, $q(W_2)=1$;
- $\mathrm{Kod}(W_3)=2$, $p_g(W_3)=1$, $q(W_3)=1$; the branch locus of the cover $V\to W_3$ is an union of four (-2)-curves.

This is an example for Propositions 5.1.1, b), 5.1.3, b) and 5.2.1.

We use Notations 4.1.1 and 6.0.2.

Step 1: *Construction of S.*
Let Q_1 be a general curve of type $5(1,(2,2)_T^3)$ (there is a pencil of such curves, see Section 4.1) and Q_2 be a general curve of type $3(1,(1,1)_T^3)$, both through points $p_0,\dots,p_3$.

Let
$$Q_1':=\widetilde{Q_1}-\sum_1^3(2E_i+2E_i')-E_0\equiv 5\widetilde{T}-\sum_1^3(2E_i+2E_i')-E_0,$$
$$Q_2':=\widetilde{Q_2}-\sum_1^3(E_i+E_i')-E_0\equiv 3\widetilde{T}-\sum_1^3(E_i+E_i')-E_0.$$
Consider the double cover $\psi:Z\to X$ with branch locus
$$\widetilde{T_1}+\dots+\widetilde{T_4}-\sum_1^4 2E_i'-4E_0,$$

where T_4 is a general line through p_0. Notice that ψ is determined by $l \equiv 2\widetilde{T} - \sum_1^3 E_i' - 2E_0$.

Let

$$\Gamma := \frac{1}{2}\psi^*(\widetilde{T_4} - E_0),\ \Gamma_i := \frac{1}{2}\psi^*(\widetilde{T_i} - E_0),$$

$$C_0 := \psi^*(E_0),$$

$$e_i := \frac{1}{2}\psi^*(E_i - E_i'),$$

$$e_i' := \psi^*(E_i') \quad i = 1, 2, 3.$$

Set:

$$\begin{aligned} &D_1 := \psi^*(Q_1') \equiv 4C_0 + 10\Gamma - \textstyle\sum_1^3(4e_i + 4e_i'),\\ &D_2 := \psi^*(Q_2') \equiv 2C_0 + 6\Gamma - \textstyle\sum_1^3(2e_i + 2e_i'),\\ &D_3 := 0 \end{aligned}$$

and

$$\begin{aligned} &L_1 := C_0 + 3\Gamma - \textstyle\sum_1^3(e_i + e_i'),\\ &L_2 := 2C_0 + 5\Gamma - \textstyle\sum_1^3(2e_i + 2e_i'),\\ &L_3 := 3C_0 + 8\Gamma - \textstyle\sum_1^3(3e_i + 3e_i'). \end{aligned}$$

Let V be the bidouble cover of Z determined by the curves D_i and by the divisors L_i and let S be the minimal model of V. Since Z is a ruled surface with $q = 1$,

$$K_Z \equiv -2C_0 - 2\Gamma + \sum_1^3 \left[(e_i + e_i') + e_i\right].$$

Therefore,

$$\begin{aligned} &K_Z + L_1 = -C_0 + \Gamma + \textstyle\sum_1^3 e_i,\\ &K_Z + L_2 = 3\Gamma - \textstyle\sum_1^3 e_i',\\ &K_Z + L_3 = C_0 + 6\Gamma - \textstyle\sum_1^3(e_i + 2e_i'). \end{aligned}$$

One has

$$\chi(\mathcal{O}_S) = 4 \cdot 0 + \frac{1}{2}\sum_1^3 L_i(K_Z + L_i) = 0 + 0 + 1 = 1.$$

Let

$$G = \widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3} - \sum_1^3(E_i + E_i') + E_4 - 2E_0.$$

As $K_Z + L_3 \equiv \psi^*(G)$ and $G - l \equiv \widetilde{T} - \sum_1^3 E_i$, then

$$h^0(Z, \mathcal{O}_Z(K_Z + L_3)) = h^0(X, \mathcal{O}_X(G)) + h^0(X, \mathcal{O}_X(G - l)) = 1 + 0 = 1.$$

This way,

$$p_g(S) = \sum_1^3 h^0(Z, \mathcal{O}_Z(K_Z + L_i)) = 0 + 0 + 1 = 1.$$

As

$$\begin{aligned} h^0(Z, \mathcal{O}_Z(2K_Z + L_1 + L_2)) &= 0, \\ h^0(Z, \mathcal{O}_Z(2K_Z + L_1 + L_3)) &> 0, \\ h^0(Z, \mathcal{O}_Z(2K_Z + L_2 + L_3)) &> 0, \end{aligned}$$

the bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, corresponding to L_1, L_2, L_3.

Step 2: *Calculation of* K_S^2.
We have

$$N := 2K_Z + \sum_1^3 L_i \equiv N_1 + N_2,$$

where

$$N_1 := 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 - \sum_1^3 2e_i'$$

and

$$N_2 := 2C_0 + 6\Gamma - \sum_1^3 (2e_i + 2e_i') \equiv \psi^* \left(3\widetilde{T} - \sum_1^3 (E_i + E_i') - E_0 \right).$$

As the support of the pullback of N_1 is a disjoint union of $24 = -N_1^2$ (-1)-curves, $|N_2|$ has no fixed component and $N_1N_2 = 0$, then

$$K_S^2 = N^2 + (-N_1^2) = N_2^2 = 4.$$

Step 3: *The surface* W_1.
Let W_1 be the double cover of Z with branch locus $D_2 + D_3$, determined by L_1. It is a ruled surface with $q = 1$ (because $q(Z) = q(S) = 1$).

Step 4: *The surface* W_2.
Let W_2 be the double cover of Z with branch locus $D_1 + D_3$, determined by L_2. As $q(Z) = q(S) = 1$, $h^0(Z, \mathcal{O}_Z(K_Z + L_2)) = 0$ and the support of

$$2(K_Z + L_2) \equiv 2\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 - 2\sum_1^3 e_i'$$

is a disjoint union of (-1)-curves, then $q(W_2) = 1$, $p_g(W_2) = 0$ and $\mathrm{Kod}(W_2) = 0$.

Step 5: *The surface* W_3.
Let W_3 be the double cover of Z with branch locus $D_1 + D_2$, determined by L_3. As $q(Z) = q(S) = 1$ and $h^0(Z, \mathcal{O}_Z(K_Z + L_3)) = 1$, then $p_g(W_3) = q(W_3) = 1$. Since $2(K_X + L_3)^2 = -10$, we get, by contraction of the 12 (-1)-curves contained in the pullback of $\widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3}$, $K^2_{W_3'} = 2$, where W_3' is the minimal model of W_3. Therefore W_3 is of general type.

6.12 $K^2 = 8,\ g = 4$, non-Du Val double plane

This section describes the steps to obtain an equation of a double plane S of general type with $K^2 = 8$, $p_g = q = 1$ and $g = 4$ such that the bicanonical map of S is not composed with the associated involution. The corresponding Magma computations are in Appendix A.4.5.

We keep Notations 4.1.1 and 6.0.2.

Let $\overline{B'}$ be a reduced curve of type $21(9, (6,6)^5_T)$, *i.e.* a reduced plane curve of degree 21 with a 9-uple point at p_0 and a $(6,6)$-point at p_i tangent to the line T_i through p_0, p_i, $i = 1, \ldots, 5$. Let $\mu : X \to \mathbb{P}^2$ be the blow-up as in Notation 6.0.2 and

$$B := \mu^*\left(\overline{B'} + \sum_1^5 T_i\right) - 14E_0 - \sum_1^5 (6E_i + 8E_i').$$

Let S be the minimal model of the double cover of X with branch locus B and i be the corresponding involution of S. One has $K_S^2 = 8$ and $\chi(\mathcal{O}_S) = 1$. From Proposition 2.1.2, a) we get $h^0(X, \mathcal{O}_X(2K_X + L)) = 1$, where L is the line bundle such that $2L \equiv B$, thus the bicanonical map of S is not composed with i.

Let us see that such a curve $\overline{B'}$ exists (hence also S exists) and that $p_g(S) = 1$. First we find points $p_0, \ldots, p_5 \in \mathbb{P}^2$ such that there exists a curve C_1 of type

$$10\left(4, (3,3)^4_T, (3,2)_T\right).$$

For this we use the methods of Section 4.2. Then we verify the existence of a cubic C_2 of type $3\left(1, 1, (1,1)^4_T\right)$, containing $p_0, \ldots, p_5$. The curve $\overline{B'}$ is an element of the pencil generated by $2C_1 + T_5$ and $6C_2 + 3T_1$. This pencil lifts to the (genus 4) Albanese fibration of S.

One has

$$K_X + L \equiv \sum_1^{10} A_i + \left(5\widetilde{T} - E_0 - \sum_1^5 (2E_i + E_i')\right),$$

where A_i, $i = 1, \ldots, 10$, are the (-2)-curves contained in $\widetilde{T_1} + \cdots + \widetilde{T_5}$. Using the Magma procedure *LinSys* (defined in Appendix A.2) we see that

$$h^0\left(X, \mathcal{O}_X\left(K_X + L - \sum_1^{10} A_i\right)\right) = 1.$$

Since $(K_X + L)A_i = -1$, $i = 1, \ldots, 10$, each A_i is fixed in $|K_X + L|$, thus $p_g(S) = h^0(X, \mathcal{O}_X(K_X + L)) = 1$.

All these calculations are done in Appendix A.4.5, using Magma.

Appendix A

Magma computations

In this appendix several computations are done using the *Computational Algebra System MAGMA* (version V2.11-14).

On the web page
http://magma.maths.usyd.edu.au
one can read
'Magma is a large, well-supported software package designed to solve computationally hard problems in algebra, number theory, geometry and combinatorics. It provides a mathematically rigorous environment for computing with algebraic, number-theoretic, combinatoric and geometric objects.'

More information about Magma can be found on the Magma on-line help system
http://magma.maths.usyd.edu.au/magma/htmlhelp/MAGMA.htm .

In Magma, a line preceded by > means an input line, something preceded by // means a comment and the symbol \ at the end of a line means continuation in the next line. The other lines are output ones.

There are two functions of Magma, which are used several times in the next sections, that deserve some words of explanation.

· **ResolutionGraph(C,p)**
'Calculate a transverse resolution graph of the plane curve singularity of C at the point p',
as one can read on the Magma online help system. For instance, the resolution of a $(3,3)$-point on a plane curve:

```
> A<x,y> := AffineSpace(Rationals(),2);
```

```
> C := Curve(A,x*((x-1)^2+y^2-1)*((x+1)^2+y^2-1));
> ResolutionGraph(C,Origin(A));
The resolution graph on the Digraph
Vertex  Neighbours

1 ([ -2, 3, 1, 0 ])     2 ;
2 ([ -1, 6, 2, 3 ])     ;
```

The first two columns indicate the self-intersection and the multiplicity of the exceptional curve, respectively. The third column refers to the canonical class and the fourth to the transverse intersection of the strict transform of C with the exceptional curve.

· **PointsOverSplittingField(Z)**
On the Magma online help system:
'If Z is a cluster [zero-dimensional scheme] this will determine some (not necessarily optimal) point set Z(L) in which all points of Z having coordinates in an algebraic closure of the base field lie and will return all points of Z(L).'

The following functions will be useful:

```
function D(F,i);
  P:=Parent(F);
  return Derivative(F,P.i);
end function;

function D2(F,i,j);
  P:=Parent(F);
  return Derivative(Derivative(F,P.i),P.j);
end function;

function D3(F,i,j,g);
  P:=Parent(F);
  return Derivative(Derivative(Derivative(F,P.i),P.j),P.g);
end function;
```

i.e. from now on
D(F,i) means Derivative(F,P.i),

D2(F,i,j) means Derivative(Derivative(F,P.i),P.j) and
D3(F,i,j,g) means Derivative(Derivative(Derivative(F,P.i),P.j),P.g).

A.1 $K^2 = 6$, $\phi_2(S)$ birational to a $K3$

The computations of Section 3.2.2 are as follows:

Step 1:

```
> K<e>:=CyclotomicField(6);//e denotes the 6th root of unity.
> //We choose a conic C with equation x1x3-x2^2=0 and fix the
> //p_i's: (1:1:1), (e^2:e:1), (e^4:e^2:1), (e^6:e^3:1),
> //(e^8:e^4:1), (e^10:e^5:1).
> R<z,s,x1,x2,x3>:=PolynomialRing(K,[3,1,1,1,1]);
> g:=&*[e^(2*i)*x1-2*e^i*x2+x3:i in [0..5]];
> //g is the product of the defining polynomials
> //of the tangent lines l_i to C at p_i.
> X:=z^2-g;
> X eq (z+x1^3-x3^3)*(z-x1^3+x3^3)+4*(x1*x3-4*x2^2)^2*\
> (-x1*x3+x2^2);//The decomposition AB+DE.
true
> i:=Ideal([s*(z-x1^3+x3^3)-4*(x1*x3-4*x2^2)^2,\
> s*(x1*x3-x2^2)-(z+x1^3-x3^3)]);
> j:=EliminationIdeal(i,1);
> j;
Ideal of Graded Polynomial ring of rank 5 over K
Lexicographical Order Variables: z, s, x1, x2, x3
Variable weights: 3 1 1 1 1 Basis:
[-1/2*s^2*x1*x3+1/2*s^2*x2^2+s*x1^3-s*x3^3+2*x1^2*x3^2-
16*x1*x2^2*x3+32*x2^4]
> 2*Basis(j)[1];
-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-
32*x1*x2^2*x3+64*x2^4
> //This is the equation of the Kummer Q.
```

Step 2:

```
> K<e>:=CyclotomicField(6);
```

```
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> Q:=Scheme(P3,F);//The Kummer.
> SQ:=SingularSubscheme(Q);
> T1:=Scheme(P3,x1-2*x2+x3); T2:=Scheme(P3,s);
> N:=Difference((T1 join T2) meet SQ, T1 meet T2);
> s:=SetToSequence(RationalPoints(N));
> //s is the sequence of the 8 nodes.
> L:=LinearSystem(P3,2);
>//This will give the h_i's:
> LinearSystem(L,[P3!s[i] : i in [1..8]]);
Linear system on Projective Space of dimension 3
  Variables: s, x1, x2, x3 with 3 sections:

s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2
s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2
s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2
```

Step 3:

```
> R<s,b,c,x1,x2,x3>:=PolynomialRing(Rationals(),6);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> h1:=s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2;
> h2:=s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2;
> h3:=s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2;
> H:=h1+b*h2+c*h3;
> I:=ideal<R|[F,H]>;
> I1:=EliminationIdeal(I,1);
> q0:=Evaluate(Basis(I1)[1],x3,1);//We work in the affine plane.
> R4<B,C,X1,X2>:=PolynomialRing(Rationals(),4);
> h:=hom<R->R4|[0,B,C,X1,X2,0]>;
> q:=h(q0);
> A4:=AffineSpace(R4);
> Sch:=Scheme(A4,[q,D(q,3),D(q,4),D2(q,3,3),D2(q,3,4),\
> D2(q,4,4),D3(q,3,3,3),D3(q,3,3,4),D3(q,3,4,4),D3(q,4,4,4)]);
> Dimension(Sch);
```

```
0
> PointsOverSplittingField(Sch);
```

This last command gives the points of Sch, as well as the necessary equations to define the field extensions where their coordinates belong. There are various solutions. One of them gives the desired quadruple point. The confirmation is as follows:

Step 4:

```
> R<x>:=PolynomialRing(Rationals());
> K<r13>:=ext<Rationals()|x^4 + x^3 + 1/4*x^2 + 3/32>;
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-\
> 32*x1*x2^2*x3+64*x2^4;
> b:=64/55*r13^3-272/55*r13^2-96/55*r13-46/55;
> c:=-2176/605*r13^3+448/605*r13^2+624/605*r13-361/605;
> H:=(s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2)+\
> b*(s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2)+\
> c*(s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2);
> Q:=Scheme(P3,F);
> C:=Scheme(Q,H);
> IsReduced(C);
false
> RC:=ReducedSubscheme(C);
> #SingularPoints(RC);//# means "number of".
1
> HasSingularPointsOverExtension(RC);
false
> pt:=Representative(SingularPoints(RC));
> pt in SingularSubscheme(Q);//pt is not a node of Q.
false
> T:=DefiningPolynomial(TangentSpace(Q,pt));
> T2:=Scheme(Q,T^2);
> #RationalPoints(T2 meet C);
1
> pt in RationalPoints(T2 meet C);
```

```
true
> HasPointsOverExtension(T2 meet C);
false
```

This way $T2$ and C generate a pencil of curves with a quadruple base point. The curve $\overline{B'}$ is a general element of this pencil.

Step 5:

```
> QA:=AffinePatch(Q,4);
> p:=Representative(RationalPoints(AffinePatch(Cluster(pt),4)));
> A3<x,y,z>:=Ambient(QA);
> psi:=map<A3->A3|[x-p[1],y-p[2],z-p[3]]>;
> Q0:=psi(QA);
> FA:=DefiningPolynomial(Q0);
> j:=[Evaluate(Derivative(FA,A3.i),Origin(A3)):i in [1,2,3]];
> J:=LinearSystem(A3,[j[1]*x+j[2]*y+j[3]*z,x^2,x*y,x*z,y^2,y*z,\
> z^2]);
> P6:=ProjectiveSpace(K,6);
> tau:=map<A3->P6|Sections(J)>;
> Degree(tau(Q0));
12
```

A.2 The procedures *LinSys* and *LinSys*2

Magma has a function which calculates linear systems of plane curves with ordinary singularities, but we want to work with non-ordinary singularities. To achieve this, we define two procedures. The first one, *LinSys*, calculates the linear system L of plane curves of degree d, in an affine plane $\mathbb{A}$, having singular points p_i of order $(m_i, m2_i)$ with tangent direction given by td_i. To define *LinSys* run the following lines:

```
procedure LinSys(A,d,p,m,m2,td,~L)
  //p,m,... are tuples of p_i,m_i,...
  x:=A.1;y:=A.2;//The coordinates of A.
  L:=LinearSystem(LinearSystem(A,d),p,m);
  for j:=1 to #m2 do
    if #Sections(L) eq 0 then break;end if;
```

```
    a:=p[j][1];b:=p[j][2];
    Bup:=[Evaluate(Sections(L)[i],y,(x-a)*y+b) div (x-a)^m[j]:\
    i in [1..#Sections(L)]];//The strict transform of the
    //blown-up curves.
    L1:=LinearSystem(A,Bup);
    L2:=LinearSystem(L1,A![a,td[j]],m2[j]);//Imposing the
    //infinitely neighbor singularity.
    if #Sections(L2) eq 0 then L:=L2;break;end if;
    Bdn:=[Evaluate((x-a)^m[j]*Sections(L2)[i],y,(y-b)/(x-a)):\
    i in [1..#Sections(L2)]];//The blown-down curves.
    R:=Universe(Bdn);
    //R is a Rational function field. We need an homomorphism
    //to send the elements of Bdn into a polynomial ring.
    h:=hom<R->CoordinateRing(A)|[x,y]>;
    L:=LinearSystem(A,[h(Bdn[i]):i in [1..#Bdn]]);
  end for;
end procedure;
```

The other procedure, *LinSys2*, calculates the sub-system J, of a given linear system L of plane curves, of those sections which have a singularity at a point q of type $m = (m_1, \dots, m_j)$ with tangent directions given by $td = [td_1, \dots, td_{j-1}]$.

```
procedure LinSys2(A,L,q,m,td,~J)
  x:=A.1;y:=A.2;//The coordinates of A.
  J:=LinearSystem(L,q,m[1]);
  td:=[q[2]] cat td;
  for j:=1 to #td-1 do
    if #Sections(J) eq 0 then break;end if;
    b:=td[j];
    Bup:=[Evaluate(Sections(J)[i],y,(x-q[1])*y+b) div\
    (x-q[1])^m[j]:i in [1..#Sections(J)]];
    J1:=LinearSystem(A,Bup);
    J:=LinearSystem(J1,A![q[1],td[j+1]],m[j+1]);
  end for;
  //
  for j:=#td-1 to 1 by -1 do
    if #Sections(J) eq 0 then break;end if;
    b:=td[j];
```

```
    Bdn:=[Evaluate((x-q[1])^m[j]*Sections(J)[i],y,(y-b)/\
    (x-q[1])):i in [1..#Sections(J)]];
    R:=Universe(Bdn);
    h:=hom<R->CoordinateRing(A)|[x,y]>;
    J:=LinearSystem(A,[h(Bdn[i]):i in [1..#Bdn]]);
  end for;
end procedure;
```

A.3 Useful curves

Here we construct some curves that are useful in constructions of Chapter 6. We use the procedure *LinSys*, defined in Appendix A.2. Recall Notation 4.1.1.

Consider, in a affine plane $\mathbb{A}$, the points

$$p_0 := (0,0),\ p_1 := (2,2),\ p_2 := (-2,2),\ p_3 := (3,1),\ p_4 := (-3,1).$$

From Section 4.1, there exists a pencil of curves of type $6(2,(2,2)^4_T)$, with singularities at $p_0,\dots,p_4$, respectively. Let G be the element of this pencil which contains the point $p_5 := (0,5)$.

· The curve G is reduced and the tangent line to G at p_5 is horizontal:

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![2,2],A![-2,2],A![3,1],A![-3,1],A![0,5],Origin(A)];
> d:=6;m:=[2,2,2,2,1,2];m2:=[2,2,2,2,1];
> td:=[p[i][2]/p[i][1]:i in [1..4]] cat [0];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
> G:=Curve(A,Sections(L)[1]);
> IsReduced(G);
true
```

· There exists a reduced curve C of type $8(4,(2,2)^4_T,(3,3))$, singular at $p_0,\dots,p_5$, such that the $(3,3)$-point is tangent to G. Moreover, $G+C$ is an reduced element of a pencil of curves of type $14(6,(4,4)^4_T,(4,4))$:

```
> d:=8;m:=[2,2,2,2,3,4];m2:=[2,2,2,2,3];
```

```
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
> C:=Curve(A,Sections(L)[1]);
> IsReduced(G join C);
true
> d:=14;m:=[4,4,4,4,4,6];m2:=[4,4,4,4,4];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
2
Scheme over Rational Field defined by
1
> #Sections(LinearSystem(L,G join C));
1
```

Analogously, one can verify that:

· there exists a reduced curve of type $7(3,(2,2)^4_T,(2,2))$, singular at $p_0,\dots,p_5$, such that the $(2,2)$-point is tangent to G.

Finally we will see that p_5 can be chosen such that

· there exist reduced curves C_1 of type $7(3,(2,2)^4_T,3)$ and C_2 of type $6(2,(2,2)^4_T,1)$, both through $p_0,\dots,p_5$, such that C_1+C_2 is reduced and the singularity of C_1+C_2 at p_5 is ordinary.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![2,2],A![-2,2],A![3,1],A![-3,1],Origin(A)];
> d:=7;m:=[2,2,2,2,3];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
6
Scheme over Rational Field defined by
1
```

Now we impose a triple point to the elements of L. This is done by asking for annulation of minors of a matrix of derivatives.

```
> R<x,y,n>:=PolynomialRing(Rationals(),3);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> H:=h(Sections(L));
> M:=[[H[i],D(H[i],1),D(H[i],2),D2(H[i],1,1),D2(H[i],1,2),\
> D2(H[i],2,2)]:i in [1..#H]];
> Mt:=Matrix(M);min:=Minors(Mt,#H);
> A:=AffineSpace(R);
> S:=Scheme(A,min cat [x-3,1+n*(y-x)*(y+x)*(3*y-x)*(3*y+x)]);
> //The condition 1+n*(..)=0 guarantees that
> //the solution is not in p.
> Dimension(S);
0
> PointsOverSplittingField(S);
```

We choose one of the solutions and show that it works:

```
> R<r1>:=PolynomialRing(Rationals());
> K<r1>:=NumberField(r1^2 - 1761803/139426560*r1 + \
> 1387488001/33730073395200);
> A<x,y>:=AffineSpace(K,2);
> y1:=-33462374400/102856069*r1 + 419793163/102856069;
> p:=[A![2,2],A![-2,2],A![3,1],A![-3,1],A![3,y1],Origin(A)];
> d:=7;m:=[2,2,2,2,3,3];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);#Sections(L);
1
> C1:=Curve(A,Sections(L)[1]);
> IsOrdinarySingularity(C1,p[5]);
true
> d:=6;m:=[2,2,2,2,1,2];m2:=[2,2,2,2];
> LinSys(A,d,p,m,m2,td,~L);#Sections(L);
1
> C2:=Curve(A,Sections(L)[1]);
> IsReduced(C1 join C2);
true
> IsSingular(C2,p[5]);
false
> IsOrdinarySingularity(C1 join C2,p[5]);
```

```
true
```

The verification that the singularities are no worst than stated is left to the reader (use the Magma functions *SingularPoints*, *HasSingularPointsOverExtension* and *ResolutionGraph*). To verify the non-existence of singularities at infinity, proceed as in the end of Appendix A.4.3.

The calculations of Section 6.8, Step 6 are as follows:

```
> d:=7;m:=[2,2,2,2,2,1,3];m2:=[2,2,2,2];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
5
Scheme over K defined by
1
> P4:=ProjectiveSpace(K,4);
> tau:=map<A->P4|Sections(L)>;
> Degree(tau(Scheme(A,Sections(L)[3])));
7
```

thus an hyperplane section of the image of τ is of degree 7.

A.4 Double planes

A.4.1 $K^2 = 8, 6, 4, 2$ and $g = 5, 4, 3, 2$

Here are the computations announced in section 4.2.1, with details for the case of a double plane with $p_g = q = 1$ and $K^2 = 6$.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![7/5,4/5],A![7/5,-4/5],A![2,1],A![2,-1],Origin(A)];
> d:=6;m:=[2,2,2,2,2];m2:=[2,2,1,1];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
5
Scheme over Rational Field defined by
1
```

We are using symmetry: we want to find $p_5 = (x, y)$ and $p_6 = (x, -y)$.

```
> R<x,y,b,c,d,e,n>:=PolynomialRing(Rationals(),7);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> l:=h(Sections(L));
> F:=l[1]+b*l[2]+c*l[3]+d*l[4]+e*l[5];
> G:=Evaluate(F,y,-y);
> //
> eqF:=2*x*y*D2(F,1,2)+x^2*D2(F,1,1)+y^2*D2(F,2,2);
> //The condition eqF=0 forces the double point p_5=(x,y)
> //to have one branch tangent to the line T_5.
> eqG:=Evaluate(eqF,y,-y);//The same to the point p_6=(x,-y).
> //
> dif:=y*(y-1/2*x)*(y-4/7*x)*(y+1/2*x)*(y+4/7*x);
```

In order to obtain $p_5, p_6 \notin T_i$, $i = 1, \dots, 4$, and $p_5 \neq p_6$, we need dif to be *different* from zero. This is achieved by imposing the condition $1 + n \cdot dif = 0$.

```
> A:=AffineSpace(R);
> Sch:=Scheme(A,[(x-2)^2+y^2-1,F,D(F,1),D(F,2),G,D(G,1),D(G,2),\
> eqF,eqG,1+n*dif]);
> Dimension(Sch);
0
> PointsOverSplittingField(Sch);
```

This command gives the points of Sch and the necessary field extensions to define them. Choosing one of the solutions we obtain the points p_5, p_6. Asking Magma for the pencil $\overline{f_A}$, we find a reduced curve B_0 of type $15(7,(4,4)^5_T,4)$. The branch locus $\overline{B} := B_0 + \sum_1^5 T_i$ is of type $20(12,(5,5)^5_T,4)$. The corresponding minimal double plane is a surface of general type with $p_g = q = 1$, $K^2 = 6$ and $g = 4$.

Verification that B_0 is as stated:

```
> R<r3>:=PolynomialRing(Rationals());
> K<r3>:=NumberField(r3^4 - 570063504574501/8986626*r3^2+\
> 1946769931994914550851531410 01/323037787455504);
> x1:=-1225449/218906496039245*r3^2 + 6763320857703/401161\
> 4254780;
> y1:=-7879209182423568/19711509531436237701627614 95*r3^3 +
> 37270947258282841632/11728154656652726662 4785*r3;
```

```
> //
> A<x,y>:=AffineSpace(K,2);
> p:=[A![7/5,4/5],A![7/5,-4/5],A![2,1],A![2,-1],A![x1,y1],\
> A![x1,-y1],Origin(A)];
> d:=15;m:=[4,4,4,4,4,4,7];m2:=[4,4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
2
Scheme over K defined by
1
> B0:=Curve(A,Sections(L)[1]+Sections(L)[2]);
> IsReduced(B0);
true
> Multiplicity(B0,Origin(A));
7
> IsOrdinarySingularity(B0,Origin(A));
true
> [Multiplicity(B0,p[i]):i in [1..6]];
[ 4, 4, 4, 4, 4, 4 ]
> IsOrdinarySingularity(B0,p[6]);
true
> T:=[Curve(A,y-p[i][2]/p[i][1]*x):i in [1..5]];
> [IntersectionNumber(B0,T[i],p[i]):i in [1..5]];
[ 8, 8, 8, 8, 8 ]
> ResolutionGraph(B0,p[1]);
The resolution graph on the Digraph
Vertex      Neighbours

1 ([ -2, 4, 1, 0 ]) 2 ;
2 ([ -1, 8, 2, 4 ]) ;
```

One obtains the same resolution graph for $p_2, ..., p_5$.

The singularities of a general element of L are no worst than the ones above, hence we do not need to verify that B_0 has no other singularities.

The other cases, $p_g = q = 1$, $K^2 = 8, 4, 2$ and $g = 5, 3, 2$, are analogous to the previous one. One needs only to ask Magma (using the procedure *LinSys*) for curves of type

$$[10 + 2i](2i + 2, (5, 5)_T^i, 4^{6-i}), \quad i = 6, 4, 3,$$

with singularities at the previous points $p_0, \ldots, p_6$. We leave this to the reader.

A.4.2 $K^2 = 7, 5, 3$ and $g = 5, 4, 3$

Here we have the detailed computations of section 4.2.2.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![1,0],A![1/5,2/5],A![2/5,1/5],A![8/5,9/5],\
> A![9/5,8/5],Origin(A)];
> d:=15;m:=[4,4,4,4,4,7];m2:=[4,4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
8
Scheme over Rational Field defined by
1
> Bup:=[Evaluate(Sections(L)[i],y,(x-1)*y+0) div (x-1)^4:\
> i in [1..#Sections(L)]];
> L:=LinearSystem(A,Bup);
> Bup:=[Evaluate(Sections(L)[i],y,(x-1)*y+0) div (x-1)^4:\
> i in [1..#Sections(L)]];
> L1:=LinearSystem(A,Bup);
```

At this stage we have imposed the $(4, 4)$-points and resolved p_1.

```
> R<x,y,u,v,n>:=PolynomialRing(Rationals(),5);
> h:=hom<PolynomialRing(L1)->R|[x,y]>;
> l:=h(Sections(L1));
```

Now we impose, to the elements of l, the necessary conditions in order to obtain the $(3, 3)$-point $p_6 = (u, v)$. The matrix Mt defined by these conditions cannot have maximal rank.

```
> H:=[Evaluate(l[i],[u,v,u,v,n]):i in [1..#l]];
> F:=[Evaluate(l[i],y,(x-u)*y+v):i in [1..#l]];
```

```
> G:=[(F[i]-Evaluate(F[i],x,u)) div (x-u):i in [1..#l]];
> G1:=[(G[i]-Evaluate(G[i],x,u)) div (x-u):i in [1..#l]];
> G2:=[(G1[i]-Evaluate(G1[i],x,u)) div (x-u):i in [1..#l]];
> F:=G2;
> M:=[[H[i],D(H[i],3),D(H[i],4),D2(H[i],3,3),D2(H[i],3,4),\
> D2(H[i],4,4),F[i],D(F[i],1),D(F[i],2),D2(F[i],1,1),\
> D2(F[i],1,2),D2(F[i],2,2)]:i in [1..#l]];
> ME:=[[Evaluate(M[i][o],[1,y,1,v,n]):o in [1..12]]:\
> i in [1..#M]];
> //This last step is needed to increase speed of calculations.
> Mt:=Matrix(ME);
> min:=Minors(Mt,#l);
> A:=AffineSpace(R);
> Sch:=Scheme(A,min cat [x-1,u-1,1+n*v]);
> Dimension(Sch);
0
> PointsOverSplittingField(Sch);
```

As before, this gives various solutions. We choose one who works. Here goes the verifications:

```
> R<r2>:=PolynomialRing(Rationals());
> K<r2>:=NumberField(r2^2 - 1292/35);
> A<x,y>:=AffineSpace(K,2);
> p:=[A![1/5,2/5],A![2/5,1/5],A![8/5,9/5],A![9/5,8/5],\
> Origin(A)];
> d:=15;m:=[4,4,4,4,7];m2:=[4,4,4,4];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> //
> q:=A![1,0];m:=[4,4,3,3];
> td:=[0,35/1292*r2,-455/15504*r2 + 455/7752];
> LinSys2(A,L,q,m,td,~J);
> #Sections(J);
1
> B0:=Curve(A,Sections(J)[1]);
> IsReduced(B0);
```

```
true
> Degree(B0);
15
> Insert(~p,1,q);
> T:=[Curve(A,y-p[i][2]/p[i][1]*x):i in [1..5]];
> [Multiplicity(B0,p[i]):i in [1..6]];
[ 4, 4, 4, 4, 4, 7 ]
> [IntersectionNumber(T[i],B0,p[i]):i in [1..5]];
[ 8, 8, 8, 8, 8 ]
> [ResolutionGraph(B0,p[i]):i in [1,7,2]];
[
    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -2, 8, 2, 1 ]) 2 3 ;
    2 ([ -2, 4, 1, 0 ]) ;
    3 ([ -2, 11, 3, 0 ])        4 ;
    4 ([ -1, 14, 4, 3 ])        ;


    ,
    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -1, 7, 1, 7 ]) ;


    ,
    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -2, 4, 1, 0 ]) 2 ;
    2 ([ -1, 8, 2, 4 ]) ;
]
```

The resolution graphs for the points p_3, p_4 and p_5 are equal to this last one.

Now we calculate the pencil which induces the Albanese fibration.

```
> d:=16;m:=[4,4,4,4,8];m2:=[4,4,4,4];
```

```
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> q:=A![1,0];m:=[4,4,4,4];
> td:=[0,35/1292*r2,-455/15504*r2 + 455/7752];
> LinSys2(A,L,q,m,td,~J);
> #Sections(J);BaseComponent(J);
2
Scheme over K defined by
1
> Jy:=LinearSystem(J,Curve(A,y));
> B0 eq Curve(A,Sections(Jy)[1] div y);
true
```

With this we have constructed a minimal double plane with $p_g = q = 1$, $K^2 = 7$ and $g = 5$.

A.4.3 $K^2 = 6$ and $g = 3$

Here we give the detailed computations of Section 4.2.3.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![4/5,7/5],A![-4/5,7/5],A![1,2],A![-1,2],Origin(A)];
> d:=6;m:=[2,2,2,2,2];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
2
Scheme over Rational Field defined by
1
> R<x,y,b,n>:=PolynomialRing(Rationals(),4);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> l:=h(Sections(L));
> F:=l[1]+b*l[2];
> G:=Evaluate(F,x,-x);
> C:=x^2+(y-2)^2-1;
> //
> eqF:=D(C,1)*D(F,2)-D(C,2)*D(F,1);//To obtain a curve
> //tangent to the conic C at p_5=(x,y).
```

```
> eqG:=Evaluate(eqF,x,-x);//The same to p_6=(-x,y).
> dif:=x*(y-2*x)*(y-7/4*x)*(y+2*x)*(y+7/4*x);
> //We need dif to be non-zero.
> //
> A:=AffineSpace(R);
> Sch:=Scheme(A,[C,F,G,eqF,eqG,1+n*D(F,1)*D(F,2)*dif]);
> Dimension(Sch);
0
> PointsOverSplittingField(Sch);
{@ (-r2, 0, 5377/5292, -3/307328),\
(r2, 0, 5377/5292, -3/307328) @}
Algebraically closed field with 2 variables
Defining relations: [
    r2^2 + 3,
    r1^2 + 3 ]
```

We found the points p_5, p_6. Now we ask for $\overline{B'} =: B0$:

```
> R<r2>:=PolynomialRing(Rationals());
> K<r2>:=NumberField(r2^2 + 3);
> A<x,y>:=AffineSpace(K,2);
> p:=[A![4/5,7/5],A![-4/5,7/5],A![1,2],A![-1,2],\
> A![-r2,0],A![r2,0],Origin(A)];
> C:=Curve(A,x^2+(y-2)^2-1);
> TangentSpace(C,p[5]);TangentSpace(C,p[6]);
Curve over K defined by -2*r2*x - 4*y + 6
Curve over K defined by 2*r2*x - 4*y + 6
> slp5:=-2*r2/4;slp6:=2*r2/4;
> //
> td:=[p[i][2]/p[i][1]:i in [1..4]] cat [slp5,slp6];
> d:=6;m:=[2,2,2,2,1,1,2];m2:=[2,2,2,2,1,1];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
> f1:=Sections(L)[1];
> //
> d:=8;m:=[2,2,2,2,2,2,4];m2:=[2,2,2,2,2,2];
> LinSys(A,d,p,m,m2,td,~L);
```

```
> #Sections(L);BaseComponent(L);
2
Scheme over K defined by
1
> #Sections(LinearSystem(L,Curve(A,f1)));
1
```

Notice that L lifts to the Albanese fibration.

```
> f2:=Sections(L)[1]+Sections(L)[2];
> B0:=Curve(A,f1*f2);
> IsReduced(B0);
true
> Multiplicity(B0,Origin(A));
6
> IsOrdinarySingularity(B0,Origin(A));
true
> [Multiplicity(B0,p[i]):i in [1..6]];
[ 4, 4, 4, 4, 3, 3 ]
> T:=[Curve(A,y-p[i][2]/p[i][1]*x):i in [1..4]];
> [IntersectionNumber(B0,T[i],p[i]):i in [1..4]];
[ 8, 8, 8, 8 ]
> ResolutionGraph(B0,p[1]);
The resolution graph on the Digraph
Vertex      Neighbours

1 ([ -2, 4, 1, 0 ]) 2 ;
2 ([ -1, 8, 2, 4 ]) ;
```

One obtains the same resolution graph for p_2, p_3 and p_4.

```
> ResolutionGraph(B0,p[5]);
The resolution graph on the Digraph
Vertex      Neighbours

1 ([ -2, 3, 1, 0 ]) 2 ;
2 ([ -1, 6, 2, 3 ]) ;
```

One obtains the same resolution graph for p_6.

We verify that the curve given by f_1 has no other singularities in $\mathbb{A}$:

```
> C:=Curve(A,f1);
> SingularPoints(C);
{@ (0, 0), (-1, 2), (-4/5, 7/5), (4/5, 7/5), (1, 2) @}
> HasSingularPointsOverExtension(C);
false
```

and that there are no singularities at infinity:

```
> PB0:=ProjectiveClosure(B0);
> Dimension(SingularSubscheme(PB0) meet LineAtInfinity(A));
-1
```

A.4.4 $K^2 = 8$ and $g = 4$

This section contains the computations of Section 4.2.4.

```
> R<r>:=PolynomialRing(Rationals());
> K<r>:=NumberField(r^2 - 53/485);
> A<x,y>:=AffineSpace(K,2);
> p:=[A![7/5,4/5],A![7/5,-4/5],A![2,1],A![2,-1],\
> A![50/97,-r],A![50/97,r],Origin(A)];
> LinearSystem(LinearSystem(A,2),[p[i]:i in [1..6]]);
Linear system on Affine Space of dimension 2
Variables : x, y  with 1 section:

-3*x + 5*y^2 + 1

> //Thus the points are contained in a conic.
> F:=x^3 - 633/97*x*y^2 - 315/97*x + 1050/97*y^2 + 70/97;
> //F gives a cubic tangent to the T_i's at the p[i]'s,
> //i=1,...,6.
> d:=4;m:=[1,1,1,1,1,1,2];m2:=[1,1,1,1,1,1];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
```

The remaining verifications are left to the reader.

A.4.5 $K^2 = 8,\ g = 4$, non-Du Val double plane

The detailed calculations of Section 6.12 are as follows:

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![1,0],A![2,1],A![1,2],A![8/5,9/5],Origin(A)];
> d:=10;m:=[3,3,3,3,4];m2:=[3,3,3,3];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);BaseComponent(L);
8
Scheme over Rational Field defined by
1
```

Now we look for the $(3,2)$-point $p_5 =: (u,v)$:

```
> R<n,x,y,u,v>:=PolynomialRing(Rationals(),5);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> l:=h(Sections(L));
> H:=[Evaluate(l[i],[n,u,v,u,v]):i in [1..#l]];
> F:=[Evaluate(l[i],y,(x-u)*y+v):i in [1..#l]];
> G:=[(F[i]-Evaluate(F[i],x,u)) div (x-u):i in [1..#l]];
> G1:=[(G[i]-Evaluate(G[i],x,u)) div (x-u):i in [1..#l]];
> G2:=[(G1[i]-Evaluate(G1[i],x,u)) div (x-u):i in [1..#l]];
> F:=G2;
> M:=[[H[i],D(H[i],4),D(H[i],5),D2(H[i],4,4),D2(H[i],4,5),\
> D2(H[i],5,5),F[i],D(F[i],2),D(F[i],3)]:i in [1..#l]];
> ME:=[[Evaluate(M[i][o],x,u):o in [1..#M[1]]]:i in [1..#M]];
> Mt:=Matrix(ME);
> min:=Minors(Mt,#l);
> A:=AffineSpace(R);
> dif:=v*(u-2*v)*(2*u-v)*(9*u-8*v);
> S:=Scheme(A,min cat [(u-1)^2+(v-1)^2-1,u*y-v,x-u,1+n*dif]);
> Dimension(S);
0
>
> PointsOverSplittingField(S);
```

We verify that the solution works:

```
> R<r1>:=PolynomialRing(Rationals());
> K<r1>:=NumberField(r1^4 - 6452/2005*r1^3 + \
> 627046/218545*r1^2 - 39636/43709*r1 + 3645/43709);
> A<x,y>:=AffineSpace(K,2);
> u1:=-5070244/445797*r1^3 + 133654601/4457970*r1^2 - \
> 33576941/2228985*r1 + 70335/33022;
> //
> p:=[A![1,0],A![2,1],A![1,2],A![8/5,9/5],A![u1,r1],A![0,0]];
> d:=10;m:=[3,3,3,3,3,4];m2:=[3,3,3,3,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
> f1:=Sections(L)[1];
> d:=3;m:=[1,1,1,1,1,1];m2:=[0,1,1,1,1];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
> f2:=Sections(L)[1];
> L:=LinearSystem(A,[f1^2*(u1*y-r1*x),f2^6*y^3]);
> BaseComponent(L);
Scheme over K defined by
1
> B1:=Curve(A,Sections(L)[1]+Sections(L)[2]);
> IsReduced(B1);
true
> Degree(B1);
21
> T:=[Curve(A,p[i][1]*y-p[i][2]*x):i in [1..5]];
> [Multiplicity(B1,p[i]):i in [1..6]];
[ 6, 6, 6, 6, 6, 9 ]
> IsOrdinarySingularity(B1,p[6]);
true
> [IntersectionNumber(T[i],B1,p[i]):i in [1..5]];
[ 12, 12, 12, 12, 12 ]
```

```
> [ResolutionGraph(B1,p[i]):i in [1..5]];

    The resolution graph on the Digraph
    Vertex      Neighbours

    1 ([ -1, 12, 2, 6 ])        2 ;
    2 ([ -2, 6, 1, 0 ]) ;
```

The points $p_2, \ldots, p_5$ have resolution graph equal to this one. Finally we check that $p_g(S) = 1$.

```
> d:=5;m:=[2,2,2,2,2,1];m2:=[1,1,1,1,1];
> LinSys(A,d,p,m,m2,td,~L);
> #Sections(L);
1
```

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