On the homotopy type of free group character varieties[1]

Ana Casimiro
Departamento de Matemática, Faculdade de Ciências e Tecnologia,
Universidade Nova de Lisboa,
Quinta da Torre, 2829-516 Caparica, Portugal
e-mail: `amc@fct.unl.pt`

Carlos Florentino
Departamento de Matemática, Instituto Superior Técnico,
Universidade de Lisboa,
Avenida Rovisco Pais, 1049-001 Lisboa, Portugal
e-mail: `cfloren@math.tecnico.ulisboa.pt`

Sean Lawton
Department of Mathematical Sciences, George Mason University
4400 University Drive, Fairfax, Virginia, 22030, USA
e-mail: `slawton3@gmu.edu`

André Oliveira
Departamento de Matemática,
Universidade de Trás-os-Montes e Alto Douro,
Quinta dos Prados, 5001-801 Vila Real, Portugal
e-mail: `agoliv@utad.pt`

Abstract Let G be a real reductive algebraic group with maximal compact subgroup K, and let F_r be a rank r free group. Here, we summarize the construction of a natural strong deformation retraction from the space of closed orbits in $\mathsf{Hom}(F_r, G)/G$ to the orbit space $\mathsf{Hom}(F_r, K)/K$. In particular, these spaces have the same homotopy type.

keywords: Character varieties; Real reductive groups; Free group representations.

1 Introduction

In this article, we present one of the main results in [CFLO], about the homotopy equivalence between two related moduli spaces of representations of a free group F_r on r generators. These are the G-character variety $\mathfrak{X}_r(G)$, consisting of the closed orbits in the conjugation quotient space $\mathsf{Hom}(F_r, G)/G$, where G is a real reductive group, and the related quotient space $\mathfrak{X}_r(K) = \mathsf{Hom}(F_r, K)/K$, where K is a maximal compact subgroup of G. One application of this result was the computation of the Poincaré polynomials of some of these character varieties, for non-compact G. We also studied the relation between the topology and *geometry* of the character varieties $\mathfrak{X}_r(G)$ and (the real points

[1]A.C., C.F. and A.O. were partially supported by FCT projects PTDC/MAT/120411/2010, MAT-GEO/0675/2012 and the Reseacrh Unit PEst-OE/MAT/UI4080/2011, Portugal; S.L. was partially supported by U.S. NSF grants DMS 1107452, 1107263, 1107367 - Gear Network, and 1309376, and the Simons Foundation grant 245642.

of) $\mathfrak{X}_r(\mathbf{G})$, where $\mathbf{G}$ is the complexification of G, making explicit use of *trace coordinates*. We provided a detailed analysis of some examples (real forms G of $\mathbf{G} = \mathrm{SL}(2, \mathbb{C})$), showing how the geometry of these spaces compare, and how to understand the deformation retraction in these coordinates. We also briefly described the Kempf-Ness sets for some of these examples. We thank the referee for the comments leading to improvements of this paper.

2 Complex and real character varieties

Let F_r be a rank r free group ($r \in \mathbb{N}$) and $\mathbf{G}$ be a complex reductive algebraic group defined over $\mathbb{R}$. We also assume $\mathbf{G}$ irreducible. The $\mathbf{G}$*-representation variety of* F_r is defined as $\mathfrak{R}_r(\mathbf{G}) := \mathsf{Hom}(F_r, \mathbf{G})$. There is a homeomorphism between $\mathfrak{R}_r(\mathbf{G})$ and $\mathbf{G}^r$, given by the evaluation map which is defined over $\mathbb{R}$, if $\mathfrak{R}_r(\mathbf{G})$ is endowed with the compact-open topology (as defined on a space of maps, with F_r given the discrete topology, and $\mathbf{G}$ the Euclidean topology from some affine embedding $\mathbf{G} \subset \mathbb{C}^n$, $n \in \mathbb{N}$), and $\mathbf{G}^r$ with the product topology. As $\mathbf{G}$ is a smooth affine variety defined over $\mathbb{R}$, $\mathfrak{R}_r(\mathbf{G})$ is also a smooth affine variety and it is defined over $\mathbb{R}$. Consider now the action of $\mathbf{G}$ on $\mathfrak{R}_r(\mathbf{G})$ by conjugation. This defines an action of $\mathbf{G}$ on the algebra $\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]$ of regular functions on $\mathfrak{R}_r(\mathbf{G})$. Let $\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]^{\mathbf{G}}$ denote the subalgebra of $\mathbf{G}$-invariant functions. Since $\mathbf{G}$ is reductive the affine categorical quotient may be defined as $\mathfrak{X}_r(\mathbf{G}) := \mathfrak{R}_r(\mathbf{G})/\!/\mathbf{G} = \mathrm{Spec}_{\max}(\mathbb{C}[\mathfrak{R}_r(\mathbf{G})]^{\mathbf{G}})$. This is a singular affine variety (irreducible and normal, since $\mathbf{G}^r$ is smooth and irreducible[2]), whose points correspond to the Zariski closures of the orbits. Since $\mathfrak{X}_r(\mathbf{G})$ is an affine variety, it is a subset of an affine space, and inherits the Euclidean topology. With respect to this topology, in [FL], it is shown that $\mathfrak{X}_r(\mathbf{G})$ is homeomorphic to the conjugation orbit space of closed orbits (called the *polystable quotient*). $\mathfrak{X}_r(\mathbf{G})$, together with that topology, is called the $\mathbf{G}$*-character variety*.

Let us define the conditions on a real Lie group G, for which our results will apply.

Definition 2.1. Let K be a compact Lie group. We say that G is a *real* K*-reductive Lie group* if the following conditions hold: (1) K is a maximal compact subgroup of G; (2) there exists a complex reductive algebraic group $\mathbf{G}$, defined over $\mathbb{R}$, such that $\mathbf{G}(\mathbb{R})_0 \subseteq G \subseteq \mathbf{G}(\mathbb{R})$, where $\mathbf{G}(\mathbb{R})$ denotes the real algebraic group of $\mathbb{R}$-points of $\mathbf{G}$, and $\mathbf{G}(\mathbb{R})_0$ its identity component (in the Euclidean topology); (3) G is Zariski dense in $\mathbf{G}$.

We note that, if $G \neq \mathbf{G}(\mathbb{R})$, then G is not necessarily an algebraic group (consider for example $G = \mathrm{GL}(n, \mathbb{R})_0$). When K is understood, we often simply call G a real reductive Lie group. All classical real matrix groups, as well as all complex reductive Lie groups, are in this setting. As an example which is not under the conditions of Definition 2.1, we can consider $\widetilde{\mathrm{SL}(n, \mathbb{R})}$, the universal covering group of $\mathrm{SL}(n, \mathbb{R})$. For $n \geqslant 2$ it is not a matrix group, and so does not satisfy our definition, since all real reductive K-groups are linear.

As above, let K be a compact Lie group, and G be a real K-reductive Lie group. In like fashion, we define the G*-representation variety of* F_r: $\mathfrak{R}_r(G) := \mathsf{Hom}(F_r, G)$. Again, $\mathfrak{R}_r(G)$ is homeomorphic to G^r. Similarly, as a set, we define $\mathfrak{X}_r(G) := \mathfrak{R}_r(G)/\!/G$ to be the set of closed orbits under the conjugation action of G on $\mathfrak{R}_r(G)$. We give $\mathfrak{X}_r(G)$ the

[2] In Subsection 2.2 of [CFLO] it was accidentally stated that $\mathfrak{X}_r(\mathbf{G})$ is not necessarily irreducible; this is generally the case when F_r is replaced by a finitely generated group Γ.

quotient topology on the subspace of points with closed orbits in $\mathfrak{R}_r(G)$. This quotient coincides with the one considered by Richardson-Slodowy in [RS, Section 7], and it is a non-trivial result in [RS] that this quotient is always Hausdorff. It is likewise called the *G-character variety of F_r* even though it may not even be a semi-algebraic set. However, it is an affine real semi-algebraic set when G is real algebraic. For K a compact Lie group, with its usual topology, we also define the space $\mathfrak{X}_r(K) := \mathsf{Hom}(F_r, K)/K \cong K^r/K$, called the *$K$-character variety* of F_r. Since the K-orbits are always closed and K is real algebraic, this construction is a special case of the construction above. So it is Hausdorff and can be identified with a semi-algebraic subset of $\mathbb{R}^d$, for some d. Moreover, it is compact, being the compact quotient of a compact space.

3 Cartan decomposition and deformation to the maximal compact

Let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{g}^{\mathbb{C}}$ the Lie algebra of $\mathbf{G}$. We will fix a Cartan involution of $\mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution, θ, of $\mathfrak{g}$. This choice allows for a Cartan decomposition of those Lie algebras. In paricular, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\theta|_{\mathfrak{k}} = 1$ and $\theta|_{\mathfrak{p}} = -1$. Furthemore, $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup, K, of G. The Cartan involution of $\mathfrak{g}$ lifts to a Lie group involution $\Theta : G \to G$ whose differential is θ and such that $K = \mathrm{Fix}(\Theta) = \{g \in G : \Theta(g) = g\}$. The multiplication map provides a diffeomorphism $G \simeq K \times \exp(\mathfrak{p})$. In particular, the exponential is injective on $\mathfrak{p}$. If we write $g = k\exp(X)$, for some $k \in K$ and $X \in \mathfrak{p}$, then $\Theta(g)^{-1}g = \exp(2X)$. So define $(\Theta(g)^{-1}g)^t := \exp(2tX)$, for any real parameter t.

Proposition 3.1. *The map $H : [0,1] \times G \to G$, $H(t,g) = f_t(g) := g(\Theta(g)^{-1}g)^{-t/2}$ is a strong deformation retraction from G to K, and for each t, f_t is K-equivariant with respect to the action of conjugation of K in G.*

By Proposition 3.1, there is a K-equivariant strong deformation retraction from G to K, so there is a K-equivariant strong deformation retraction from G^r onto K^r with respect to the diagonal action of K. This immediately implies:

Corollary 3.2. *Let K be a compact Lie group and G be a real K-reductive Lie group. Then $\mathfrak{X}_r(K)$ is a strong deformation retract of $\mathfrak{R}_r(G)/K$.*

4 Kempf-Ness set and deformation retraction for character varieties

As before, fix a compact Lie group K, and a real K-reductive Lie group G. Suppose that G acts linearly on a complex vector space $\mathbb{V}$, equipped with a Hermitian inner product $\langle\,,\rangle$. Without loss of generality we can assume that $\langle\,,\rangle$ is K-invariant, by averaging.

Definition 4.1. A vector $X \in \mathbb{V}$ is a *minimal vector* for the action of G on $\mathbb{V}$ if $\|X\| \leqslant \|g \cdot X\|$, for every $g \in G$, where $||\cdot||$ is the norm corresponding to $\langle\,,\rangle$.

Let $\mathcal{KN}_G = \mathcal{KN}(G, \mathbb{V})$ denote the set of minimal vectors. $\mathcal{KN}_G$ is known as the *Kempf-Ness set* in $\mathbb{V}$ with respect to the action of G. It is a closed algebraic set in $\mathbb{V}$.

The Kempf-Ness theory also works for closed G-subspaces. Indeed, let Y be an arbitrary closed G-invariant subspace of $\mathbb{V}$, and define $\mathcal{KN}_G^Y := \mathcal{KN}_G \cap Y$. The next theorem is proved in [RS, Proposition 7.4, Theorems 7.6, 7.7 and 9.1].

Theorem 4.2. *The quotient $Y/\!\!/G$ is a closed Hausdorff space and if, in addition, Y is a real algebraic subset of $\mathbb{V}$, it is also homeomorphic to a closed semi-algebraic set in some $\mathbb{R}^d$. Moreover, there is a K-equivariant deformation retraction of Y onto $\mathcal{KN}_G^Y$.*

To apply the Kempf-Ness theorem to our situation, we need to embed the G-invariant closed set $Y = \mathfrak{R}_r(G) = \mathsf{Hom}(F_r, G) \cong G^r$ in a complex vector space $\mathbb{V}$, as follows. As $\mathbf{G}$ is a complex reductive algebraic group, $\mathbf{G}$ and G can embedded in some $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{R})$, respectively. In this case, the Cartan involution is given by $\theta(A) = -A^t$, so that $\Theta(g) = (g^{-1})^t$. From now on, we will assume this situation. We can obtain the embedding of K^r ($r \in \mathbb{N}$) into the vector space given by the product of the spaces of all n-square complex matrices, which we denote by $\mathbb{V}$. Precisely, $\mathbb{V} := \mathfrak{gl}(n,\mathbb{C})^r \cong \mathbb{C}^{rn^2}$. The adjoint representation of $\mathrm{GL}(n,\mathbb{C})$ in $\mathfrak{gl}(n,\mathbb{C})$ restricts to a representation $G \to \mathrm{Aut}(\mathbb{V})$ given by $g \cdot (X_1, \ldots, X_r) = (gX_1g^{-1}, \ldots, gX_rg^{-1}),\ g \in G,\ X_i \in \mathfrak{gl}(n,\mathbb{C})$. Moreover, this yields a representation $\mathfrak{g} \to \mathrm{End}(\mathbb{V})$ of the Lie algebra $\mathfrak{g}$ of G in $\mathbb{V}$ given by the Lie brackets: $A \cdot (X_1, \ldots, X_r) = (AX_1 - X_1A, \ldots, AX_r - X_rA) = ([A, X_1], \ldots, [A, X_r])$ for every $A \in \mathfrak{g}$ and $X_i \in \mathfrak{gl}(n,\mathbb{C})$. In what follows, the context will be clear enough to distinguish the notations of the above representations. We choose an inner product $\langle\, ,\rangle$ in $\mathfrak{gl}(n,\mathbb{C})$ which is K-invariant, under the restriction of the representation $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{gl}(n,\mathbb{C}))$ to K. From this we obtain an inner product on $\mathbb{V}$, K-invariant by the corresponding diagonal action of K: $\langle (X_1, \ldots, X_r), (Y_1, \ldots, Y_r)\rangle = \sum_{i=1}^r \langle X_i, Y_i\rangle$ for $X_i, Y_j \in \mathfrak{gl}(n,\mathbb{C})$. In $\mathfrak{gl}(n,\mathbb{C})$, $\langle\, ,\rangle$ can be given explicitly by $\langle A, B\rangle = \mathrm{tr}(A^*B)$. So by Theorem 4.2, Proposition 3.1 and Corollary 3.2 we have the following theorem:

Theorem 4.3. *The spaces $\mathfrak{X}_r(G)$ and $\mathfrak{X}_r(K)$ have the same homotopy type. In particular, the homotopy type of $\mathfrak{X}_r(G)$ depends only on the maximal compact subgroup K of G.*

In our setting, the Kempf-Ness can be explicitly described as the closed set given by:

Proposition 4.4. $\mathcal{KN}_G^Y = \left\{(g_1, \cdots, g_r) \in G^r :\ \sum_{i=1}^r g_i^* g_i = \sum_{i=1}^r g_i g_i^*\right\}$. *In particular, we have the inclusion $K^r \cong \mathsf{Hom}(F_r, K) \subset \mathcal{KN}_G^Y$.*

For G algebraic, there is a natural inclusion of finite CW-complexes $\mathfrak{X}_r(K) \subset \mathfrak{X}_r(G)$ (see Lemma 4.9 of [CFLO]). Using this result, Proposition 4.4, Theorem 4.2 and Whitehead's Theorem we achieve the main result: (see Section 6 of [CFLO] for an example)

Theorem 4.5. *There is a strong deformation retraction from $\mathfrak{X}_r(G)$ to $\mathfrak{X}_r(K)$.*

References

[CFLO] A. Casimiro, C. Florentino, S. Lawton, A. Oliveira, "Topology of moduli spaces of free group representations in real reductive groups", *Forum Math*, DOI: 10.1515/forum-2014-0049.

[FL] C. Florentino and S. Lawton, "Topology of character varieties of Abelian groups", *Topology and its Applications*, Vol. 173 (2014), pp. 32-58.

[RS] R. W. Richardson, P. J. Slodowy, "Minimum vectors for real reductive algebraic groups", *J. London Math. Soc.*, (2) Vol. 42, No. 3 (1990), pp. 409-429.