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Regular Article

The magnetic field circulation counterpart to Biot-Savart's law

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Abstract. The circulation, around an arbitrarily shaped loop, of the magnetic field generated by the flow of a volume distribution of current through a conducting medium is derived, in the zero retarded-time limit, using the Biot-Savart law. This circulation formula is validated for the particular case where the conducting medium is a finite length of arbitrarily shaped wire, and the distinction between the resulting expression and one obtained from the magnetic scalar potential is highlighted.

1 Introduction

Both Biot-Savart's law and Ampère's law for magnetic field circulation (abbreviated hereafter Ampère's law) play important roles in electromagnetism. For high-symmetry problems Ampère's law offers an extraordinarily efficient way of calculating the magnetic field $\vec{B}$, the configurations that can be handled through it being the infinite straight line, the infinite plane, the infinite solenoid and the toroid [1], *i.e.* configurations for which the current has to be closed in order to satisfy the requirement of continuity (charge conservation), *e.g.* [2,3]. By contrast, Biot-Savart's law holds in circumstances more general than this, which are detailed in [4]. Suffice to add that the use of Ampère's law is restricted to the static régime (*i.e.* charge density ρ and current density $\vec{J}$ both constant in time t), whereas Biot-Savart's law is also valid, for instance, in the semistatic régime (ρ linear and $\vec{J}$ constant in t), see [4]. Consider, for example, the cases of the infinitely long tubular conductor of negligible thickness and of the filamentary wire segment. For the first case, the magnetic field can be obtained directly from Ampère's law, *e.g.* [5,6], whereas solution of the same problem using Biot-Savart's law is non-trivial. By contrast, for the second case, *i.e.* determination of the magnetic field contribution resulting from the flow of steady current through a finite length of straight wire (integrated in some circuit), the application of Ampère's law is non-trivial, whereas the use of Biot-Savart's law is direct, as detailed in Morvay and Pálfalvi's [3] analysis of the different solutions to the finite wire problem. The above two examples motivate the quest for simple expressions that combine the advantages of Ampère's law with those of Biot-Savart's, *i.e.* direct applicability in high-symmetry problems which involve determination of the magnetic field resulting from steady flow of current through a finite portion of conductor. Moreover, the finite conductor problem has been identified as one of sources of student difficulties regarding the correct application of Ampère's law [2,3,7], which is a relevant topic in physics teaching [8]. Recently, [9] used the integral form of Ampère-Maxwell's law in conjunction with Coulomb's law in the zero retarded-time limit to obtain a simple formula that allows the circulation, around an arbitrarily shaped loop, of the magnetic field generated by a current in an arbitrarily shaped finite wire, to be expressed in terms of the wire current and of the solid angles subtended by the circulation path and each of the wire's endpoints. More recently, [10, 11] used the formula in [9] to determine the magnetic fields generated by finite length conducting tubes, cylinders, and coaxial cables. While Biot-Savart's law gives the same result as the formula in [9] if applied to a finite length of straight conducting wire, there is no evidence that this is true if the wire and/or the circulation path are arbitrarily shaped.

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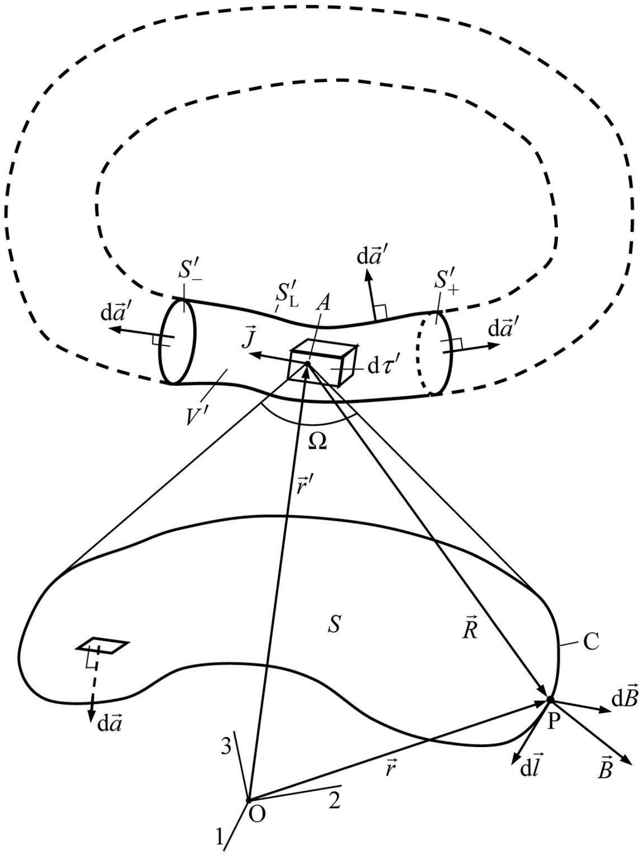

Fig. 1. Magnetic field $\vec{B}$ at P generated by conducting region of volume V' bounded by closed surface $S' = S'_+ + S'_- + S'_L$ is the sum of contributions $d\vec{B}$ at P from the current density $\vec{J}$ at each elementary volume $d\tau'$ within V'. Circulation around contour C is the sum of all $\vec{B} \cdot d\vec{l}$, where the elementary length vector $d\vec{l}$ is tangent to C. Position vectors $\vec{R}$, $\vec{r}$ and $\vec{r}'$, solid angle Ω subtended by C at point A, open surface S bounded by C, elementary area vector $d\vec{a}$ in S, elementary area vector $d\vec{a}\,'$ in closed surface S'. Conducting medium outside region of interest is between the dashed lines. Origin (O) of Cartesian coordinate system (axes 1, 2, 3).

Moreover, a simple circulation formula has yet to be derived for a volume distribution of current. The aims of the present study are: i) To derive, from Biot-Savart's law, a formula for the contribution to the circulation, around an arbitrarily shaped loop, of the magnetic field generated by a volume distribution of current through a conducting medium under zero retarded-time conditions (fig. 1); ii) To apply the analysis detailed in Morvay and Pálfalvi [3] to the circulation formula in i) with the aim of validating such formula for the particular case where the conducting medium is a thin finite wire of arbitrary shape (fig. 2); iii) To highlight the distinction between the validated result in ii) and a formula derived from the magnetic scalar potential expression in [12,13], see fig. 3.

2 Results and discussion

2.1 Circulation of the magnetic field generated by a volume distribution of current

Consider a current density distribution $\vec{J} \equiv \vec{J}(\vec{r}\,')$ within a conducting medium which includes a region of volume V' bounded by closed surface $S' = S'_+ + S'_- + S'_L$ (fig. 1), $\vec{r}\,'$ being the position vector of any point in V'. Under zero retarded-time conditions, the contribution of V' to the total magnetic field generated at any point P with position vector $\vec{r}$ and located on an arbitrarily shaped contour C that does *not* intersect V' (fig. 1), is given by Biot-Savart's law for a volume distribution as

$$\vec{B}(\mathrm{P}) = \frac{\mu_0}{4\pi} \int_{V'} \vec{J} \times \frac{\vec{R}}{R^3} d\tau', \tag{1}$$

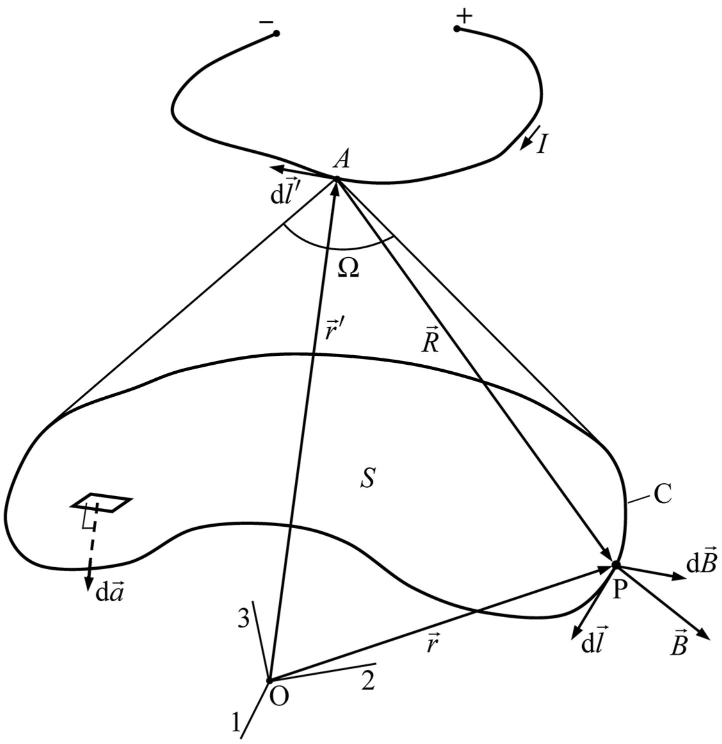

Fig. 2. Magnetic field $\vec{B}$ at P generated by current (I) through finite wire (endpoints + and −) is the sum of all contributions $\mathrm{d}\vec{B}$ at P from such wire's elementary length vectors $\mathrm{d}\vec{l}'$. Circulation around contour C is sum of all $\vec{B}\cdot\mathrm{d}\vec{l}$, with elementary length vector $\mathrm{d}\vec{l}$ tangent to C. Position vectors $\vec{R}$, $\vec{r}$ and $\vec{r}'$, solid angle Ω subtended by C at point A, open surface S bounded by C, elementary area vector $\mathrm{d}\vec{a}$ in S. Origin (O) of Cartesian coordinate system (axes 1, 2, 3).

where $\mathrm{d}\tau'$ is an elementary volume element centered at $\vec{r}'$, and $\vec{R}$ the position vector of P relative to $\mathrm{d}\tau'$, *i.e.*

$$\vec{R} = \vec{r} - \vec{r}', \tag{2}$$

the integration in (1) being over the primed coordinates. The magnetic field circulation around contour C in fig. 1 can then be expressed, using (1) and simple vector identities, as

$$\begin{aligned}\oint_{\mathrm{C}} \vec{B}\cdot\mathrm{d}\vec{l} &= \frac{\mu_0}{4\pi}\oint_{\mathrm{C}} \mathrm{d}\vec{l}\cdot\int_{V'} \vec{J}\times\frac{\vec{R}}{R^3}\mathrm{d}\tau' \\ &= \frac{\mu_0}{4\pi}\int_{V'} \vec{J}\,\mathrm{d}\tau'\cdot\oint_{\mathrm{C}} \mathrm{d}\vec{l}\times\frac{(-\vec{R})}{R^3}\,,\end{aligned} \tag{3}$$

where $\mathrm{d}\vec{l}$ is the elementary length vector tangent to C at point P. The second integral in the last equality of (3) can be determined using the mathematical technique applied in problem 5.1 of [13] to obtain the magnetic scalar potential formula given therein and used later in this study. For the sake of completeness this technique is detailed in the appendix, resulting in

$$\oint_{\mathrm{C}} \mathrm{d}\vec{l}\times\frac{(-\vec{R})}{R^3} = \vec{\nabla}'\Omega, \tag{4}$$

where the prime denotes that the differentiation in nabla is with respect to source coordinates $\vec{r}'$, Ω is the solid angle apexed at A and subtended by contour C (fig. 1), *i.e.*

$$\Omega = \int_S \frac{\vec{R}\cdot\mathrm{d}\vec{a}}{R^3}\,, \tag{5}$$

S being any open surface bounded by C and that does not intersect apex A (*i.e.* $\vec{R}\neq\vec{0}$), and elementary area vector $\mathrm{d}\vec{a}$ in S is positively oriented with respect to the circulation of $\mathrm{d}\vec{l}$ around C. Inserting (4) into (3) gives the contribution

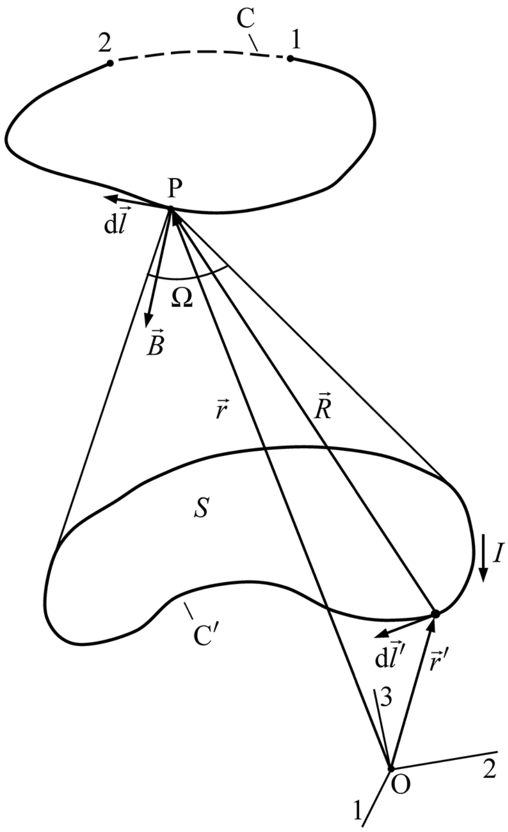

Fig. 3. Magnetic field $\vec{B}$ generated by *closed* loop C′ of current (I), at any point P on a path (1 to 2) *not* intersecting C′ is $\vec{B} = (\mu_0 I/4\pi)\vec{\nabla}\Omega$, see [12,13]. Differentiation in $\vec{\nabla}$ is with respect to field coordinates ($\vec{r}$). In [12] points 1 and 2 spatially coincide, so that line integral of $\vec{B}$ from 1 to 2 becomes circulation around a closed contour (C).

to magnetic field circulation resulting from the flow of current in the conducting region of volume V' as

$$\oint_{\mathrm{C}} \vec{B}\cdot \mathrm{d}\vec{l} = \frac{\mu_0}{4\pi}\int_{V'} \mathrm{d}\tau' \vec{J}\cdot\left(\vec{\nabla}'\Omega\right) \tag{6a}$$

or, equivalently, using simple vector identities in conjunction with Gauss's divergence theorem in (6a),

$$\begin{aligned}\oint_{\mathrm{C}} \vec{B}\cdot \mathrm{d}\vec{l} &= \frac{\mu_0}{4\pi}\left[\int_{S'} \Omega\vec{J}\cdot \mathrm{d}\vec{a}\,' - \int_{V'} \mathrm{d}\tau'\Omega\vec{\nabla}'\cdot\vec{J}\right]\\ &= \mu_0\int_{S'}\frac{\Omega}{4\pi}\vec{J}\cdot \mathrm{d}\vec{a}\,' + \mu_0\frac{\partial}{\partial t}\int_{V'}\rho\,\mathrm{d}\tau'\frac{\Omega}{4\pi}\\ &= \mu_0\int_{S'}\frac{\Omega}{4\pi}\vec{J}\cdot \mathrm{d}\vec{a}\,' + \mu_0\frac{\partial}{\partial t}\int_{Q'}\frac{\Omega}{4\pi}\mathrm{d}q'\end{aligned} \tag{6b}$$

where the second equality resulted from application of the continuity equation, $\rho \equiv \rho(\vec{r}\,', t)$ being the charge density, Q' the total charge within S', and $\mathrm{d}q'$ the elementary charge within $\mathrm{d}\tau'$. Equation (6), which is the main result of our study, is the circulation counterpart to Biot-Savart's law for a volume distribution of current, and likewise only valid under zero retarded-time conditions. In the following subsection, (6) is validated for the particular case where the current distribution is an arbitrarily shaped thin wire.

2.2 Application to the arbitrarily shaped finite wire problem

Application of magnetic field circulation formula (6b) to this problem requires making the region of volume V' in fig. 1 sufficiently thin as to approximate an arbitrarily shaped wire such as that in fig. 2, and using the reasoning outlined in Morvay and Pálfalvi [3] for a straight wire of finite length. Very briefly, these authors argued that the finite wire has to be integrated in some circuit to account for charge conservation, and that there are two possible approaches to determine the finite length straight wire's contribution to magnetic field circulation. *Approach 1*: to replace the remainder of the circuit by an incoming current at one wire end and an outgoing current at the other end, which in the present context means an incoming current at S'_+ and an outgoing current at S'_- in fig. 1. In such model *no* time-varying charges are present in the region of interest, there is no displacement current, and the current *crosses* S'_+ and S'_- in fig. 1. To the best of our knowledge Morvay and Pálfalvi [3] were the first and only authors to propose and implement this approach, using an ingenious method outlined in their work. *Approach 2*: to replace the remainder of the circuit by placing two opposite time-varying charges $+Q$ and $-Q$ at the locations of the discontinuity, *i.e.* S'_+ and S'_- in fig. 1. In this model the *only* time-varying charges in the region of interest are the ones above, there is a displacement current, and the current $I = -\mathrm{d}Q/\mathrm{d}t$ is *confined* to volume V'. By contrast to approach 1, approach 2 has been extensively used for a finite length straight wire, *e.g.* [2,3,7,9,14–17]. The two approaches are next applied to a volume distribution which approximates a thin arbitrarily shaped wire. If approach 1 is applied, the second term in (6b) vanishes (no time-varying charges in the region of interest) and, since in this model there is current crossing surfaces S'_- and S'_+ in fig. 1, the surviving first term of (6b) simplifies in the thin wire limit as

$$\begin{aligned}\oint_{\mathrm{C}} \vec{B}\cdot \mathrm{d}\vec{l} &= \mu_0 \int_{S'} \frac{\Omega}{4\pi} \vec{J}\cdot \mathrm{d}\vec{a}' \\ &= \mu_0 \int_{S'_-} \frac{\Omega}{4\pi} J\, \mathrm{d}a' \cos 0 + \mu_0 \int_{S'_+} \frac{\Omega}{4\pi} J\, \mathrm{d}a' \cos \pi \\ &\approx \mu_0 I \frac{\Omega^- - \Omega^+}{4\pi}\,, \end{aligned} \tag{7a}$$

where, for an arbitrarily shaped thin wire, solid angle Ω in fig. 1 becomes approximately constant at S'_- and at S'_+, taking the respective values Ω^- and Ω^+, the apex of these solid angles tending in the thin wire limit to endpoints $-$ and $+$ of fig. 2. On the other hand, if approach 2 is that which is applied, it is now the first term in (6b) that vanishes (the current is *confined* to the wire: it does not cross S') and, since in this model the only time-varying charges in the region of interest ($-Q$ and $+Q$) are confined respectively to S'_- and S'_+ in fig. 1, the surviving second term of (6b) simplifies in the thin wire limit as

$$\begin{aligned}\oint_{\mathrm{C}} \vec{B}\cdot \mathrm{d}\vec{l} &= \mu_0 \frac{\partial}{\partial t} \int_{Q'} \frac{\Omega}{4\pi} \mathrm{d}q' \\ &\approx \mu_0 \frac{\Omega_-}{4\pi} \frac{\mathrm{d}}{\mathrm{d}t}(-Q) + \mu_0 \frac{\Omega_+}{4\pi} \frac{\mathrm{d}}{\mathrm{d}t}(+Q) \\ &= \mu_0 I \frac{\Omega^- - \Omega^+}{4\pi}\,, \end{aligned} \tag{7b}$$

thereby showing that either of the aforementioned approaches gives the same result for the arbitrarily shaped thin finite wire. Moreover eq. (7) of the present study, which originated from Biot-Savart's law, is identical to eq. (6) of [9], which originated from the integral form of Ampère-Maxwell's law in conjunction with Coulomb's law, such identity being in line with the validity of all three laws under zero retarded-time conditions.

For completeness it is worth noting that, while it is mathematically simpler to obtain (7) from the Biot-Savart law in line element form, *i.e.* from $\vec{B}(\mathrm{P}) = \frac{\mu_0 I}{4\pi} \int_+^- \mathrm{d}\vec{l}' \times \frac{\vec{R}}{R^3}$ with the variables defined in fig. 2, such method misses the physical insight afforded by the previous derivation's resort to approaches 1 and 2. Suffice to add that, using the above expression, the circulation integral simplifies as

$$\begin{aligned}\oint_{\mathrm{C}} \vec{B}\cdot \mathrm{d}\vec{l} &= \frac{\mu_0 I}{4\pi} \int_+^- \mathrm{d}\vec{l}' \cdot \oint_{\mathrm{C}} \mathrm{d}\vec{l} \times \frac{(-\vec{R}\,)}{R^3} \\ &= \frac{\mu_0 I}{4\pi} \int_+^- \mathrm{d}\vec{l}' \cdot \left(\vec{\nabla}'\Omega\right) \\ &= \frac{\mu_0 I}{4\pi} \int_+^- \mathrm{d}\Omega \\ &= \mu_0 I \frac{\Omega^- - \Omega^+}{4\pi}\,, \end{aligned} \tag{7c}$$

where the second equality was obtained resorting to (4), Ω is defined in (5), and Ω^+ and Ω^- are the values of Ω at endpoints $+$ and $-$ in fig. 2. Carefully note that since, in (7c), $\mathrm{d}\Omega$ is integrated along the path from the point at

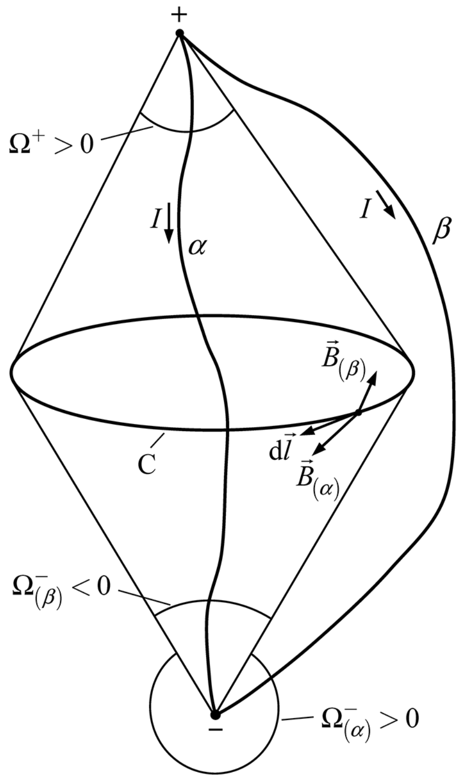

Fig. 4. The finite length wire (endpoints + and −) traversed by current (I) threads circulation contour C if it follows path α and does not if it follows β, resulting, respectively, in magnetic fields $\vec{B}_{(\alpha)}$ or $\vec{B}_{(\beta)}$. If C lies between + and − (case shown), for a set solid angle of Ω^+ at endpoint +, the solid angle at the other end (−) is $\Omega^-_{(\alpha)}$ (for path α) or $\Omega^-_{(\beta)} = -(4\pi - \Omega^-_{(\alpha)})$ (for path β). Solid angle positive if right-handed corkscrew rotating with $\mathrm{d}\vec{l}$ advances towards solid angle horizon, negative otherwise.

highest potential (labelled +) to that at lowest potential (labelled −), for a set initial value of Ω^+ the final value Ω^- depends not only on Ω^+ but also on whether the wire path does or does not thread C, consistent with [9]. Similar conclusion applies to (7a) and (7b) because Ω is continuous over the integration region. For instance, applying (7) to the example in fig. 4, if the wire threads C (path α) then $\oint_{\mathrm{C}} \vec{B}_{(\alpha)} \cdot \mathrm{d}\vec{l} = \mu_0 I(\Omega^-_{(\alpha)} - \Omega^+)/4\pi$, whereas if it does not (path β) then $\oint_{\mathrm{C}} \vec{B}_{(\beta)} \cdot \mathrm{d}\vec{l} = \mu_0 I(\Omega^-_{(\alpha)} - \Omega^+ - 4\pi)/4\pi$, $\vec{B}_{(\alpha)}$ and $\vec{B}_{(\beta)}$ being defined in fig. 4. Evidently, if the distance between endpoints + and − in fig. 3 increases to infinity, then $\Omega^+ = 0$, $\Omega^-_{(\alpha)} = 4\pi$ and $\oint_{\mathrm{C}} \vec{B} \cdot \mathrm{d}\vec{l}$ reduces to Ampère's law, *i.e.* to $\mu_0 I$ (path α) and zero (path β). The present study is concluded by highlighting the distinction between (7) and a formula which can be obtained from the magnetic scalar potential expression in [12,13].

2.3 The magnetic scalar potential formula

A cursory comparison between (7) and the equations on page 98 of [12] suggests similarities between them, thereby demanding clarification of the distinction between our work and that of [12], whose equations can be expressed in the present study's notation as

$$\vec{B} \cdot \mathrm{d}\vec{l} = \frac{\mu_0 I}{4\pi} \left(\vec{\nabla}\Omega\right) \cdot \mathrm{d}\vec{l} \tag{8}$$

$$\oint_{\mathrm{C}} \vec{B} \cdot \mathrm{d}\vec{l} = \frac{\mu_0 I}{4\pi} \oint_{\mathrm{C}} \mathrm{d}\Omega = \mu_0 I \tag{9a}$$

$$\oint_{\mathrm{C}} \vec{B} \cdot \mathrm{d}\vec{l} = 0, \tag{9b}$$

where (9a) and (9b) correspond, respectively, to circulation contour C interlocking or not with the loop (C′) of current, the latter being the case shown in fig. 3. At the outset, both in the present study (eq. (7)) and in [12] the magnetic field's line integral path never intersects the wire of current, see figs. 2 and 3. However, unlike in (7), where $\vec{B}$ is the *contribution* of the finite wire (*i.e.* of part of a circuit) to the circuit's total magnetic field, in (8) and (9) $\vec{B}$ is the magnetic field generated by closed loop C′ in fig. 3, *i.e.* by the *whole* circuit, so that the meaning of $\vec{B}$ is different in the two cases. For the case shown in fig. 3, at any point P that does not intersect C′, the magnetic field generated by C′ is [12,13]

$$\vec{B} = -\vec{\nabla}\Phi_{\mathrm{M}} = \frac{\mu_0 I}{4\pi}\vec{\nabla}\Omega, \tag{10}$$

where Φ_{M} is the magnetic scalar potential and Ω the solid angle subtended, at each point P where the magnetic field is to be determined, by the closed loop of current (fig. 3). Moreover, since (10) was obtained for a *closed* loop of current [13], its use to determine magnetic field circulation has to give the same result as Ampère's law, as is the case in [12], see (9). It is, of course, possible to instead use (10) to determine the line integral of the magnetic field generated by loop of current C′, but along an integration path which need not be closed, *e.g.* points 1 and 2 in fig. 3, giving

$$\begin{aligned}\int_1^2 \vec{B}\cdot \mathrm{d}\vec{l} &= \frac{\mu_0 I}{4\pi}\int_1^2 (\nabla\Omega)\cdot \mathrm{d}\vec{l} \\ &= \frac{\mu_0 I}{4\pi}\int_1^2 \mathrm{d}\Omega \\ &= \mu_0 I\,\frac{\Omega^{(2)}-\Omega^{(1)}}{4\pi}\,,\end{aligned} \tag{11}$$

where $\Omega^{(1)}$ and $\Omega^{(2)}$ are the solid angles subtended by loop of current C′ and, respectively, with apexes at ends 1 and 2 of the integration path. Although (11) is strikingly similar to (7) it deals with the reverse problem, *i.e.* while (7) concerns a closed integration path for the magnetic field generated by a non-closed circuit (fig. 2), (11) concerns a non-closed integration path for the magnetic field generated by a closed circuit (fig. 3), the result from the two integrals being indistinguishable on condition that the geometries of figs. 2 and 3 are identical. Of course, as highlighted earlier, the meaning of $\vec{B}$ is different in the two cases.

3 Conclusions

Biot-Savart's law for a volume distribution of current was used to obtain a circulation formula for the magnetic field in the zero retarded-time limit. The formula was applied to an arbitrarily shaped thin wire of finite length and the resulting expression found to be identical to a recent result which expressed magnetic field circulation in terms of the wire current and of the solid angles subtended by the circulation path and each of the wire endpoints. Finally, the line integral around a closed path of the magnetic field generated by a non-closed portion of circuit, was shown to be identical to that resulting from the interchange of the portion of circuit with the integration path.

Appendix A.

Noting that the second integral in the last equality of (3) is a vector quantity, its i-th Cartesian component ($i = 1, 2, 3$) can be written as

$$\begin{aligned}\left[\oint_{\mathrm{C}} \mathrm{d}\vec{l}\times\frac{(-\vec{R})}{R^3}\right]_i &= \hat{x}_i\cdot\oint_{\mathrm{C}} \mathrm{d}\vec{l}\times\frac{(-\vec{R})}{R^3} \\ &= \oint_{\mathrm{C}} \hat{x}_i\times\frac{\vec{R}}{R^3}\cdot \mathrm{d}\vec{l},\end{aligned} \tag{A.1}$$

where $\hat{x}_i$ are the unit vectors along axes 1, 2 and 3 in fig. 1. Applying Stokes' curl theorem to (A.1) gives

$$\begin{aligned}\left[\oint_{\mathrm{C}} \mathrm{d}\vec{l}\times\frac{(-\vec{R})}{R^3}\right]_i &= \int_S \left[\vec{\nabla}\times\left(\hat{x}_i\times\frac{\vec{R}}{R^3}\right)\right]\cdot \mathrm{d}\vec{a} \\ &= \int_S \left[\left(\frac{\vec{R}}{R^3}\cdot\vec{\nabla}\right)\hat{x}_i - \left(\hat{x}_i\cdot\vec{\nabla}\right)\frac{\vec{R}}{R^3} + \hat{x}_i\left(\vec{\nabla}\cdot\frac{\vec{R}}{R^3}\right) - \frac{\vec{R}}{R^3}\left(\vec{\nabla}\cdot\hat{x}_i\right)\right]\cdot \mathrm{d}\vec{a} \\ &= \int_S \left[\vec{0} - \frac{\partial}{\partial x_i}\frac{\vec{R}}{R^3} + \hat{x}_i 4\pi\delta\left(\vec{R}\right) - \vec{0}\right]\cdot \mathrm{d}\vec{a},\end{aligned} \tag{A.2}$$

where the differentiation in $\vec{\nabla}$ is with respect to the field coordinates $\vec{r}$ (*i.e.* to x_1, x_2, x_3), and S is any open surface bounded by C that does not intersect point A, *i.e.* $\vec{R} \neq \vec{0}$, see fig. 1. Therefore, (A.2) reduces to

$$\begin{aligned}\left[\oint_{\mathrm{C}} \mathrm{d}\vec{l} \times \frac{(-\vec{R})}{R^3}\right]_i &= \int_S \left(-\frac{\partial}{\partial x_i}\frac{\vec{R}}{R^3}\right) \cdot \mathrm{d}\vec{a} \\ &= \int_S \left(+\frac{\partial}{\partial x_i'}\frac{\vec{R}}{R^3}\right) \cdot \mathrm{d}\vec{a} \\ &= \frac{\partial}{\partial x_i'} \int_S \frac{\vec{R}}{R^3} \cdot \mathrm{d}\vec{a},\end{aligned} \tag{A.3}$$

where the change from field coordinates $\vec{r}$ (*i.e.* x_1, x_2, x_3) to source coordinates $\vec{r}\,'$ (*i.e.* to x_1', x_2', x_3') in the partial derivatives of the second surface integral was made using (2). Noting that in $\vec{\nabla}\,'$ the prime denotes differentiation with respect to the source coordinates, (A.3) is the i-th component of $\vec{\nabla}\,'$, so that the result given in (4) is obtained, *i.e.* $\oint_{\mathrm{C}} \mathrm{d}\vec{l} \times \frac{(-\vec{R})}{R^3} = \vec{\nabla}\,'\Omega$, where Ω is the solid angle with apex at A and subtended by contour C, given by (5) and shown in fig. 1.

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