

# HIGGS BUNDLES FOR THE NON-COMPACT DUAL OF THE UNITARY GROUP

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ABSTRACT. Using Morse-theoretic techniques, we show that the moduli space of  $U^*(2n)$ -Higgs bundles over a compact Riemann surface is connected.

## 1. INTRODUCTION

Let  $X$  be a compact Riemann surface of genus  $g \geq 2$ , and let  $\mathcal{M}_G$  be the moduli space of polystable  $G$ -Higgs bundles over  $X$ , where  $G$  is a real reductive Lie group.  $G$ -Higgs bundles for  $G = \mathrm{GL}(m, \mathbb{C})$  were introduced by Hitchin in [17]. In this case a  $G$ -Higgs bundle is a pair  $(V, \varphi)$  consisting of a holomorphic bundle  $V$  over  $X$  and a holomorphic section  $\varphi$  of the bundle  $\mathrm{End} V$  twisted with the canonical bundle of  $X$ . In this paper we study the moduli space  $\mathcal{M}_{U^*(2n)}$ , where  $U^*(2n)$  is the subgroup of  $\mathrm{GL}(2n, \mathbb{C})$  consisting of matrices  $M$  verifying that  $\overline{M}J_n = J_nM$  where  $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ . This group is the non-compact dual of  $U(2n)$  in the sense that the non-compact symmetric space  $U^*(2n)/\mathrm{Sp}(2n)$  is the dual of the compact symmetric space  $U(2n)/\mathrm{Sp}(2n)$  in Cartan's classification of symmetric spaces (cf. [16]).

The complex general linear group  $\mathrm{GL}(m, \mathbb{C})$  has as real forms the groups  $U(p, q)$ , with  $p + q = m$  (including the compact real form  $U(m)$ ), the split real form  $\mathrm{GL}(m, \mathbb{R})$  and, when  $m = 2n$  also  $U^*(2n)$ . In a similar fashion as the images in  $\mathcal{M}_{\mathrm{GL}(m, \mathbb{C})}$  of the moduli spaces  $\mathcal{M}_{U(p, q)}$  with  $p + q = m$  appear as fixed point subvarieties of  $\mathcal{M}_{\mathrm{GL}(m, \mathbb{C})}$  under the involution  $(V, \varphi) \mapsto (V, -\varphi)$ , the image of the moduli space  $\mathcal{M}_{U^*(2n)}$  in  $\mathcal{M}_{\mathrm{GL}(m, \mathbb{C})}$  is a fixed point subvariety of  $\mathcal{M}_{\mathrm{GL}(2n, \mathbb{C})}$  under the involution  $(V, \varphi) \mapsto (V^*, \varphi^t)$ , together with  $\mathcal{M}_{\mathrm{GL}(2n, \mathbb{R})}$  (see [9] and [14]). The number of connected components of  $\mathcal{M}_{U(p, q)}$  were determined in [1] and the ones of  $\mathcal{M}_{\mathrm{GL}(m, \mathbb{R})}$  in [2] and [18]. The real form  $U^*(2n)$  was

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*Date:* 10 September 2010.

*2000 Mathematics Subject Classification.* 14D20, 14F45, 14H60.

*Key words and phrases.* Semistable Higgs bundles, connected components of moduli spaces.

Authors partially supported by Ministerio de Educación y Ciencia and Conselho de Reitores das Universidades Portuguesas through Acção Integrada Luso-Espanhola n<sup>o</sup> E-38/09, Moduli of quadratic bundles (2009-2010). First author partially supported by the Spanish Ministerio de Ciencia e Innovación (MICINN) under grant MTM2007-67623. The first author thanks the Max Planck Institute for Mathematics in Bonn — that he was visiting when this paper was completed — for support. Second author partially supported by Fundação para a Ciência e Tecnologia - FCT (Portugal) through the Centro de Matemática da Universidade de Trás-os-Montes e Alto Douro - CMUTAD, and through Project PTDC/MAT/099275/2008. Both authors thank the referee for invaluable suggestions that improved the presentation and for drawing our attention to a gap in the first version of the paper.

therefore the remaining one for which the number of connected components was still to be determined. In this paper we prove the following.

**Theorem.** *The moduli space  $\mathcal{M}_{U^*(2n)}$  of  $U^*(2n)$ -Higgs bundles over  $X$  is connected.*

We adopt the Morse-theoretic approach pioneered by Hitchin in [17], and which has already been applied for several other groups (see, for example, [18, 15, 1, 2, 13, 3, 11]), to reduce our problem to the study of connectedness of certain subvarieties of  $\mathcal{M}_{U^*(2n)}$ . For that, we obtain first a detailed description of smooth points of the moduli space  $\mathcal{M}_{U^*(2n)}$ , then we give also an explicit description of stable and non-simple  $U^*(2n)$ -Higgs bundles, and show how the polystable  $U^*(2n)$ -Higgs bundles split as a direct sum of stable objects.

Non-abelian Hodge theory on  $X$  establishes a homeomorphism between  $\mathcal{M}_G$  and the moduli space of reductive representations of  $\pi_1 X$  in  $G$  (cf. [17, 29, 30, 8, 7, 10, 6]). A direct consequence of our result is thus the following.

**Theorem.** *The moduli space of reductive representations of  $\pi_1 X$  in  $U^*(2n)$  is connected.*

The connectedness of  $\mathcal{M}_{U^*(2n)}$  reflects the fact that  $U^*(2n)$  is simply-connected. It seems plausible that, like for  $\mathcal{M}_{U^*(2n)}$ ,  $\mathcal{M}_G$  is connected whenever  $G$  is a real reductive Lie group with  $\pi_1 G = 0$ . When  $G$  is complex this has been proved by Hitchin [17] for  $SL(2, \mathbb{C})$  and by Simpson [29] for  $SL(n, \mathbb{C})$ . For general complex  $G$ , the result follows from a theorem by Li [20], showing the analogous result for the moduli space of flat  $G$ -connections, and the non-abelian Hodge theory correspondence. As far as we know, there is no proof in general using Higgs bundle techniques.

## 2. $U^*(2n)$ -HIGGS BUNDLES

Let  $X$  be a compact Riemann of genus  $g \geq 2$ , and let  $G$  a real reductive Lie group. Let  $H \subseteq G$  be a maximal compact subgroup and  $H^{\mathbb{C}} \subseteq G^{\mathbb{C}}$  their complexifications. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be a Cartan decomposition of  $\mathfrak{g}$ , where  $\mathfrak{m}$  is the complement of  $\mathfrak{h}$  with respect to the non-degenerate  $\text{Ad}(G)$ -invariant bilinear form on  $\mathfrak{g}$ . If  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$  is the corresponding Cartan involution then  $\mathfrak{h}$  and  $\mathfrak{m}$  are its  $+1$ -eigenspace and  $-1$ -eigenspace, respectively. Complexifying, we have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

and  $\mathfrak{m}^{\mathbb{C}}$  is a representation of  $H^{\mathbb{C}}$  through the so-called *isotropy representation*

$$(2.1) \quad \iota : H^{\mathbb{C}} \longrightarrow \text{Aut}(\mathfrak{m}^{\mathbb{C}})$$

which is induced by the adjoint representation of  $G^{\mathbb{C}}$  on  $\mathfrak{g}^{\mathbb{C}}$ . If  $E_{H^{\mathbb{C}}}$  is a principal  $H^{\mathbb{C}}$ -bundle over  $X$ , we denote by  $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = E_{H^{\mathbb{C}}} \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$  the vector bundle, with fiber  $\mathfrak{m}^{\mathbb{C}}$ , associated to the isotropy representation.

Let  $K = T^*X^{1,0}$  be the canonical line bundle of  $X$ .

**Definition 2.1.** *Let  $X$  be a compact Riemann surface of genus  $g \geq 2$ . A  $G$ -Higgs bundle over  $X$  is a pair  $(E_{H^{\mathbb{C}}}, \varphi)$  where  $E_{H^{\mathbb{C}}}$  is a principal holomorphic  $H^{\mathbb{C}}$ -bundle over  $X$  and  $\varphi$  is a global holomorphic section of  $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$ , called the Higgs field.*

Higgs bundles were introduced by Hitchin [17] on compact Riemann surfaces and by Simpson [30] on any compact Kähler manifold.

*Example 2.2.*

- (i) If  $G$  is compact, a  $G$ -Higgs bundle is simply a holomorphic  $G^{\mathbb{C}}$ -principal bundle. For instance, a  $U(n)$ -Higgs bundle is simply a holomorphic  $GL(n, \mathbb{C})$ -principal bundle over  $X$  or, in terms of holomorphic vector bundles, a  $U(n)$ -Higgs bundle is a rank  $n$  holomorphic vector bundle.
- (ii) If  $G$  is complex with maximal compact  $H$ , we have that  $H^{\mathbb{C}} = G$  and  $\mathfrak{m} = \sqrt{-1}\mathfrak{h}$ , so  $\mathfrak{m}^{\mathbb{C}} = \mathfrak{g}$ . Thus a  $G$ -Higgs bundle is a pair  $(E_G, \varphi)$  where  $E_G$  is a holomorphic  $G$ -bundle and  $\varphi \in H^0(E_G(\mathfrak{g}) \otimes K)$  where  $E_G(\mathfrak{g})$  denotes the adjoint bundle of  $E_G$ , obtained from  $E_G$  under the adjoint action of  $G$  on  $\mathfrak{g}$ :  $E_G(\mathfrak{g}) = E_G \times_{\text{Ad}} \mathfrak{g}$ . As an example, a  $GL(m, \mathbb{C})$ -Higgs bundle is, in terms of vector bundles, a pair  $(V, \varphi)$  with  $V$  a holomorphic rank  $m$  vector bundle and  $\varphi \in H^0(\text{End}(V) \otimes K)$ .

Let us now consider the case of the real Lie group  $U^*(2n)$ . A possible way to realize the group  $U^*(2n)$  as a matrix group is as the subgroup of  $GL(2n, \mathbb{C})$  defined as

$$U^*(2n) = \{M \in GL(2n, \mathbb{C}) \mid \overline{M}J_n = J_nM\},$$

where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

From this definition, it is obvious that  $U^*(2n)$  is the real form of  $GL(2n, \mathbb{C})$  given by the fixed point set of the involution  $\sigma : GL(2n, \mathbb{C}) \rightarrow GL(2n, \mathbb{C})$ ,  $\sigma(M) = J_n^{-1}\overline{M}J_n$ .

*Remark 2.3.*  $U^*(2n)$  is also the group of quaternionic linear automorphisms of an  $n$ -dimensional vector space over the ring  $\mathbb{H}$  of quaternions, and therefore  $U^*(2n)$  is also denoted by  $GL(n, \mathbb{H})$ .

A maximal compact subgroup of  $U^*(2n)$  is the compact symplectic group  $Sp(2n)$  (or, equivalently, the group of  $n \times n$  quaternionic unitary matrices), whose complexification is  $Sp(2n, \mathbb{C})$ , the complex symplectic group.

The corresponding Cartan decomposition of the complex Lie algebras is

$$\mathfrak{gl}(2n, \mathbb{C}) = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}},$$

where  $\mathfrak{m}^{\mathbb{C}} = \{A \in \mathfrak{gl}(2n, \mathbb{C}) \mid A^t J_n = J_n A\}$ . Hence:

**Definition 2.4.** A  $U^*(2n)$ -Higgs bundle over  $X$  is a pair  $(E, \varphi)$ , where  $E$  is a holomorphic  $Sp(2n, \mathbb{C})$ -principal bundle and the Higgs field  $\varphi$  is a global holomorphic section of  $E \times_{Sp(2n, \mathbb{C})} \mathfrak{m}^{\mathbb{C}} \otimes K$ .

Now, if  $\mathbb{W}$  is the standard  $2n$ -dimensional complex representation of  $Sp(2n, \mathbb{C})$  and  $\Omega$  denotes the standard symplectic form on  $\mathbb{W}$ , then the isotropy representation space is

$$\mathfrak{m}^{\mathbb{C}} = S_{\Omega}^2 \mathbb{W} = \{\xi \in \text{End}(\mathbb{W}) \mid \Omega(\xi \cdot, \cdot) = \Omega(\cdot, \xi \cdot)\} \subset \text{End}(\mathbb{W}).$$

Given a symplectic vector bundle  $(W, \Omega)$ , denote by  $S_\Omega^2 W$  the bundle of endomorphisms  $\xi$  of  $W$  which are symmetric with respect to  $\Omega$  i.e. such that  $\Omega(\xi \cdot, \cdot) = \Omega(\cdot, \xi \cdot)$ . In the vector bundle language, we have hence the following:

**Definition 2.5.** A  $U^*(2n)$ -Higgs bundle over  $X$  is a triple  $(W, \Omega, \varphi)$ , where  $W$  is a holomorphic vector bundle of rank  $2n$ ,  $\Omega \in H^0(\Lambda^2 W^*)$  is a symplectic form on  $W$  and the Higgs field  $\varphi \in H^0(S_\Omega^2 W \otimes K)$  is a  $K$ -twisted endomorphism  $W \rightarrow W \otimes K$ , symmetric with respect to  $\Omega$ .

Given the symplectic form  $\Omega$ , we have the usual skew-symmetric isomorphism

$$\omega : W \xrightarrow{\cong} W^*$$

given by

$$\omega(v) = \Omega(v, -).$$

It follows from the symmetry of  $\varphi$  with respect to  $\Omega$  that

$$(2.2) \quad (\varphi^t \otimes 1_K)\omega = (\omega \otimes 1_K)\varphi.$$

*Remark 2.6.* Given a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$ , define the homomorphism

$$\tilde{\varphi} : W^* \longrightarrow W \otimes K$$

by

$$(2.3) \quad \tilde{\varphi} = \varphi\omega^{-1}.$$

It follows from (2.2) that  $\tilde{\varphi}$  is skew-symmetric i.e.

$$\tilde{\varphi}^t \otimes 1_K = -\tilde{\varphi}.$$

In other words,

$$\tilde{\varphi} \in H^0(\Lambda^2 W \otimes K).$$

Hence, since  $\omega : W \rightarrow W^*$  is an isomorphism, it is equivalent to think of a  $U^*(2n)$ -Higgs bundle as a triple  $(W, \Omega, \varphi)$  with  $\varphi \in H^0(S_\Omega^2 W \otimes K)$  or as a triple  $(W, \Omega, \tilde{\varphi})$  with  $\tilde{\varphi} \in H^0(\Lambda^2 W \otimes K)$ .

Given a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$ , we must then have  $W \cong W^*$ , thus

$$\deg(W) = 0.$$

In other words, the topological invariant of these objects given by the degree is always zero. This is of course consequence of the fact that the group  $U^*(2n)$  is connected and simply-connected and that, for  $G$  connected,  $G$ -Higgs bundles are topologically classified (cf. [24]) by the elements of  $\pi_1 G$ .

*Remark 2.7.* Two  $G$ -Higgs bundles  $(E_{H^c}, \varphi)$  and  $(E'_{H^c}, \varphi')$  over  $X$  are *isomorphic* if there is an holomorphic isomorphism  $f : E_{H^c} \rightarrow E'_{H^c}$  such that  $\varphi' = \tilde{f}(\varphi)$ , where  $\tilde{f} \otimes 1_K : E_{H^c}(\mathfrak{m}^c) \otimes K \rightarrow E'_{H^c}(\mathfrak{m}^c) \otimes K$  is the map induced from  $f$  and from the isotropy representation  $H^c \rightarrow \text{Aut}(\mathfrak{m}^c)$ . Hence, two  $U^*(2n)$ -Higgs bundles  $(W, \Omega, \varphi)$  and  $(W', \Omega', \varphi')$  are *isomorphic* if there is an isomorphism  $f : W \rightarrow W'$  such that  $\omega = f^t \omega' f$  and  $\varphi' f = (f \otimes 1_K)\varphi$ , which is equivalent to  $\tilde{\varphi}' = (f \otimes 1_K)\tilde{\varphi} f^t$  where  $\tilde{\varphi}$  is given in (2.3).

## 3. MODULI SPACES

**3.1. Stability conditions.** Now we consider the moduli space of  $G$ -Higgs bundles, for which we need the notions of stability, semistability and polystability.

In order to find these notions for  $U^*(2n)$ -Higgs bundles, we briefly recall here the main definitions. The main reference is [10], where the general notion of (semi,poly)stability is deduced in detail and where several examples are studied. Let  $E_{H^{\mathbb{C}}}$  be a principal  $H^{\mathbb{C}}$ -bundle. Let  $\Delta$  be a fundamental system of roots of the Lie algebra  $\mathfrak{h}^{\mathbb{C}}$ . For every subset  $A \subseteq \Delta$  there is a corresponding parabolic subgroup  $P_A \subseteq H^{\mathbb{C}}$ . Let  $\chi : \mathfrak{p}_A \rightarrow \mathbb{C}$  be an antidominant character of  $\mathfrak{p}_A$ , the Lie algebra of  $P_A$ . Let  $\sigma$  be a holomorphic section of  $E_{H^{\mathbb{C}}}(G/P_A)$ , that is, a reduction of the structure group of  $E_{H^{\mathbb{C}}}$  to  $P_A$ , and denote by  $E_{\sigma}$  the corresponding  $P_A$ -bundle. We define the *degree* of  $E_{H^{\mathbb{C}}}$  with respect to  $\sigma$  and  $\chi$  by

$$\deg E_{H^{\mathbb{C}}}(\sigma, \chi) = \deg \chi_* E_{\sigma}.$$

When  $\chi$  lifts to a character  $P_A \rightarrow \mathbb{C}^*$ , then the degree of  $E_{H^{\mathbb{C}}}$  is written in terms of the degree of the line bundle obtained from  $E_{\sigma}$  and from the character  $P_A \rightarrow \mathbb{C}^*$ . When  $\chi$  does not lift, the degree of  $E_{H^{\mathbb{C}}}$  is also the degree of a certain line bundle obtained also from  $E_{\sigma}$  and  $\chi$ . There is also an alternative definition of degree, in terms of Chern-Weil theory. The detailed definitions of degree can be found in Sections 2.3-2.6 of [10].

Let  $\iota : H^{\mathbb{C}} \rightarrow \text{Aut}(\mathfrak{m}^{\mathbb{C}})$  be the isotropy representation. We define

$$\begin{aligned} \mathfrak{m}_{\chi}^{-} &= \{v \in \mathfrak{m}^{\mathbb{C}} : \iota(e^{tsx})v \text{ is bounded as } t \rightarrow \infty\} \\ \mathfrak{m}_{\chi}^0 &= \{v \in \mathfrak{m}^{\mathbb{C}} : \iota(e^{tsx})v = v \text{ for every } t\}. \end{aligned}$$

Let  $\langle \cdot, \cdot \rangle$  be an invariant  $\mathbb{C}$ -bilinear pairing on  $\mathfrak{h}^{\mathbb{C}}$ .

Here is the general definition of semistability, given in [10]. It depends on a parameter  $\alpha \in \sqrt{-1}\mathfrak{h} \cap \mathfrak{z}$ , where  $\mathfrak{z}$  is the center of  $\mathfrak{h}^{\mathbb{C}}$ .

**Definition 3.1.** *Let  $\alpha \in \sqrt{-1}\mathfrak{h} \cap \mathfrak{z}$ . A  $G$ -Higgs bundle  $(E_{H^{\mathbb{C}}}, \varphi)$  is:*

- $\alpha$ -semistable if

$$\deg E_{H^{\mathbb{C}}}(\sigma, \chi) - \langle \alpha, \chi \rangle \geq 0,$$

*for any parabolic subgroup  $P_A$  of  $H^{\mathbb{C}}$ , any antidominant character  $\chi$  of  $\mathfrak{p}_A$  and any reduction of structure group  $\sigma$  of  $E_{H^{\mathbb{C}}}$  to  $P_A$  such that  $\varphi \in H^0(E_{\sigma}(\mathfrak{m}_{\chi}^{-}) \otimes K)$ , where  $E_{\sigma}$  is the corresponding  $P_A$ -bundle.*

- $\alpha$ -stable if it is  $\alpha$ -semistable, and

$$\deg E_{H^{\mathbb{C}}}(\sigma, \chi) - \langle \alpha, \chi \rangle > 0,$$

*for any  $P_A$ ,  $\chi$  and  $\sigma$  as above such that  $\varphi \in H^0(E_{\sigma}(\mathfrak{m}_{\chi}^{-}) \otimes K)$ ,  $A \neq \emptyset$  and  $\chi \notin (\mathfrak{h} \cap \mathfrak{z})^*$ .*

- $\alpha$ -polystable if it is  $\alpha$ -semistable, and for each  $P_A$ ,  $\sigma$  and  $\chi$  as above such that

$$\deg E_{H^{\mathbb{C}}}(\sigma, \chi) - \langle \alpha, \chi \rangle = 0,$$

*there exists a holomorphic reduction of the structure group,  $\sigma_L \in H^0(E_{\sigma}(P_A/L_A))$ , of  $E_{\sigma}$  to the Levi subgroup  $L_A$  of  $P_A$ , such that  $\varphi \in H^0(E_{\sigma_L}(\mathfrak{m}_{\chi}^0) \otimes K)$ , where  $E_{\sigma_L}$  is the corresponding  $L_A$ -bundle.*

3.1.1. *GL(n, C)-Higgs bundles.* Recall from Example 2.2 that a  $GL(n, \mathbb{C})$ -Higgs bundle is a pair  $(V, \varphi)$  where  $V$  is a rank  $n$  vector bundle and  $\varphi \in H^0(\text{End}(V) \otimes K)$ . A subbundle  $V'$  of  $V$  is said to be  $\varphi$ -invariant if  $\varphi(V') \subset V' \otimes K$ .

The *slope* of a vector bundle  $V$  is defined by the quotient  $\mu(V) = \text{deg}(V)/\text{rk}(V)$ , where  $\text{rk}(V)$  denotes the rank of  $V$ .

It can be seen that, when applied to  $GL(n, \mathbb{C})$ -Higgs bundles, the (semi,poly)stability condition of Definition 3.1, simplifies as follows:

**Proposition 3.2.** *A  $GL(n, \mathbb{C})$ -Higgs bundle  $(V, \varphi)$  is:*

- semistable if and only if  $\mu(V') \leq \mu(V)$  for every proper  $\varphi$ -invariant subbundle  $V' \subsetneq V$ .
- stable if and only if  $\mu(V') < \mu(V)$  for every proper  $\varphi$ -invariant subbundle  $V' \subsetneq V$ .
- polystable if and only if it is semistable and, for every proper  $\varphi$ -invariant subbundle  $V' \subsetneq V$  such that  $\mu(V') = \mu(V)$ , there is another proper  $\varphi$ -invariant subbundle  $V'' \subsetneq V$  such that  $\mu(V'') = \mu(V)$  and  $V = V' \oplus V''$ .

Notice that, on the last item of the previous proposition,  $(V', \varphi|_{V'})$  and  $(V'', \varphi|_{V''})$  must also be polystable (this holds due to the Hitchin-Kobayashi correspondence between polystable  $GL(n, \mathbb{C})$ -Higgs bundles and solutions to the  $GL(n, \mathbb{C})$ -Hitchin equations; see [17]). So, an iteration procedure shows that a  $GL(n, \mathbb{C})$ -Higgs bundle  $(V, \varphi)$  is polystable if and only if  $V = V_1 \oplus \cdots \oplus V_k$ , where  $\varphi(V_i) \subset V_i \otimes K$  and  $(V_i, \varphi|_{V_i})$  are stable  $GL(\text{rk}(V_i), \mathbb{C})$ -Higgs bundles with  $\mu(V_i) = \mu(V)$  (cf. [22]).

*Remark 3.3.* It can be seen that a  $GL(n, \mathbb{C})$ -Higgs bundle  $(V, \varphi)$  is  $\alpha$ -semistable if and only if  $\alpha = \mu(V)$  and  $\mu(V') \leq \mu(V)$  for all proper subbundle  $V' \subsetneq V$  such that  $\varphi(V') \subset V' \otimes K$ . So, the parameter is fixed by the topological type of the  $GL(n, \mathbb{C})$ -Higgs bundle. Hence,  $\alpha = \mu(V)$  is the value of the parameter which we are considering in the previous proposition.

3.1.2.  *$U^*(2n)$ -Higgs bundles.* Also the general definition of (semi,poly)stability for  $G$ -Higgs bundles given above, simplifies in the case  $G = U^*(2n)$ , as we shall now briefly explain. The main reference for this, and where this is done in detail for several groups, is again [10]. In order to state the stability condition for  $U^*(2n)$ -Higgs bundles, we first introduce some notation.

For any filtration of vector bundles

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

satisfying  $W_{k-j} = W_j^{\perp\Omega}$  (here  $W_j^{\perp\Omega}$  denotes the orthogonal complement of  $W_j$  with respect to  $\Omega$ ), define

$$\Lambda(\mathcal{W}) = \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i \leq \lambda_{i+1} \text{ and } \lambda_{k-i+1} = -\lambda_i \text{ for any } i\}.$$

For each  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \Lambda(\mathcal{W})$  consider the subbundle of  $S_\Omega^2 W \otimes K$  given by

$$(3.1) \quad N(\mathcal{W}, \lambda) = S_\Omega^2 W \otimes K \cap \sum_{\lambda_j \leq \lambda_i} \text{Hom}(W_i, W_j) \otimes K \subseteq S_\Omega^2 W \otimes K$$

and let

$$(3.2) \quad d(\mathcal{W}, \lambda) = \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}) \deg W_j.$$

According to [10] the stability conditions for a  $U^*(2n)$ -Higgs bundle can now be stated as follows.

**Proposition 3.4.** *A  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  is:*

- semistable if and only if  $d(\mathcal{W}, \lambda) \geq 0$  for every filtration  $\mathcal{W}$  as above and any  $\lambda \in \Lambda(\mathcal{W})$  such that  $\varphi \in H^0(N(\mathcal{W}, \lambda))$ .
- stable if and only if it is semistable and  $d(\mathcal{W}, \lambda) > 0$  for every choice of filtration  $\mathcal{W}$  and any nonzero  $\lambda \in \Lambda(\mathcal{W})$  such that  $\varphi \in H^0(N(\mathcal{W}, \lambda))$ .
- polystable if and only if it is semistable and, for every filtration  $\mathcal{W}$  as above and any  $\lambda \in \Lambda(\mathcal{W})$  satisfying  $\lambda_i < \lambda_{i+1}$  for each  $i$ , such that  $\varphi \in H^0(N(\mathcal{W}, \lambda))$  and  $d(\mathcal{W}, \lambda) = 0$ , there is an isomorphism

$$W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

such that

$$\Omega(W_i/W_{i-1}, W_j/W_{j-1}) = 0, \quad \text{unless } i + j = k + 1.$$

Furthermore, via this isomorphism,

$$\varphi \in H^0\left(\bigoplus_i \text{End}(W_i/W_{i-1}) \otimes K\right).$$

*Remark 3.5.* The center of  $\mathfrak{sp}(2n, \mathbb{C})$  is trivial hence, in the case of  $G = U^*(2n)$ , the only possibility for the value of the parameter  $\alpha$  of Definition 3.1 is  $\alpha = 0$ . So this is the value of  $\alpha$  that we are considering in the previous proposition.

There is a simplification of the stability condition for  $U^*(2n)$ -Higgs bundles analogous to the cases considered in [10]. Recall that a subbundle  $W'$  of  $W$  is  $\varphi$ -invariant if  $\varphi(W') \subset W' \otimes K$  i.e.  $\varphi|_{W'} \in H^0(\text{End}(W') \otimes K)$ .

**Proposition 3.6.** *A  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  is:*

- semistable if and only if  $\deg W' \leq 0$  for any isotropic and  $\varphi$ -invariant subbundle  $W' \subset W$ .
- stable if and only if it is semistable and  $\deg W' < 0$  for any isotropic and  $\varphi$ -invariant strict subbundle  $0 \neq W' \subset W$ .
- polystable if and only if it is semistable and, for any isotropic (resp. coisotropic) and  $\varphi$ -invariant strict subbundle  $0 \neq W' \subset W$  such that  $\deg W' = 0$ , there is another coisotropic (resp. isotropic) and  $\varphi$ -invariant subbundle  $0 \neq W'' \subset W$  such that  $W \cong W' \oplus W''$ .

*Proof.* The proof follows *mutatis mutandis* the proof of Theorem 4.4 of [10]. Take a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$ , and assume that  $\deg W' \leq 0$  for any isotropic,  $\varphi$ -invariant, subbundle  $W' \subset W$ . We want to prove that  $(W, \Omega, \varphi)$  is semistable. Suppose that  $\varphi$  is nonzero, for otherwise the result follows from the usual characterization of

(semi)stability for  $\mathrm{Sp}(2n, \mathbb{C})$ -principal bundles due to Ramanathan (see Remark 3.1 of [24]).

Choose any filtration

$$\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

satisfying  $W_{k-j} = W_j^{\perp\Omega}$  for any  $j$ , and consider the convex set

$$(3.3) \quad \Lambda(\mathcal{W}, \varphi) = \{\lambda \in \Lambda(\mathcal{W}) \mid \varphi \in H^0(N(\mathcal{W}, \lambda))\} \subset \mathbb{R}^k,$$

where  $N(\mathcal{W}, \lambda)$  is defined in (3.1).

Let

$$(3.4) \quad \mathcal{J} = \{j \mid \varphi(W_j) \subset W_j \otimes K\} = \{j_1, \dots, j_r\}.$$

One checks easily that if  $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda(\mathcal{W})$  then

$$(3.5) \quad \lambda \in \Lambda(\mathcal{W}, \varphi) \iff \lambda_a = \lambda_b \text{ for any } j_i < a \leq b \leq j_{i+1}.$$

We claim that the set of indices  $\mathcal{J}$  is symmetric:

$$(3.6) \quad j \in \mathcal{J} \iff k - j \in \mathcal{J}.$$

Checking this is equivalent to prove that  $\varphi(W_j) \subset W_j \otimes K$  implies that  $\varphi(W_j^{\perp\Omega}) \subset W_j^{\perp\Omega} \otimes K$ . Suppose that this is not true, so that for some  $j$  we have  $\varphi(W_j) \subset W_j \otimes K$  and there exists some  $w \in W_j^{\perp\Omega}$  such that  $\varphi(w) \notin W_j^{\perp\Omega} \otimes K$ . Then there exists  $v \in W_j$  such that  $\Omega(v, \varphi(w)) \neq 0$ . However, since  $\varphi$  is symmetric with respect to  $\Omega$ , we must have  $\Omega(v, \varphi(w)) = \Omega(\varphi(v), w)$ , and the latter vanishes because by assumption  $\varphi(v)$  belongs to  $W_j$ . So we have reached a contradiction, and (3.6) holds.

Let  $\mathcal{J}' = \{j \in \mathcal{J} \mid 2j \leq k\}$  and, for each  $j \in \mathcal{J}'$ , define the vector

$$L_j = - \sum_{c \leq j} e_c + \sum_{d \geq k-j+1} e_d$$

where  $e_1, \dots, e_k$  is the canonical basis of  $\mathbb{R}^k$ . It follows from (3.5) and (3.6) that  $\Lambda(\mathcal{W}, \varphi)$  is the positive span of the vectors  $\{L_j \mid j \in \mathcal{J}'\}$ . Hence,

$$(3.7) \quad d(\mathcal{W}, \lambda) \geq 0 \text{ for any } \lambda \in \Lambda(\mathcal{W}, \varphi) \iff d(\mathcal{W}, L_j) \geq 0 \text{ for any } j \in \mathcal{J}'.$$

Now, we compute  $d(\mathcal{W}, L_j) = -\deg W_{k-j} - \deg W_j$ . On the other hand, since  $W_{k-j} = W_j^{\perp\Omega}$ , we have an exact sequence  $0 \rightarrow W_{k-j} \rightarrow W \rightarrow W_j^* \rightarrow 0$  (the projection is given by  $v \mapsto \Omega(v, -)$ ), so  $0 = \deg W^* = \deg W_{k-j} + \deg W_j^*$ , hence  $\deg W_{k-j} = \deg W_j$ . Therefore  $d(\mathcal{W}, L_j) \geq 0$  is equivalent to  $\deg W_j \leq 0$ , which holds by assumption, because  $W_j$  is  $\varphi$ -invariant and isotropic for every  $j \in \mathcal{J}'$ . Hence, from (3.7) and Proposition 3.4, it follows that  $(W, \Omega, \varphi)$  is semistable.

The converse statement, namely, that if  $(W, \Omega, \varphi)$  is semistable then for any isotropic and  $\varphi$ -invariant subbundle  $W' \subset W$  we have  $\deg W' \leq 0$ , is immediate by applying the stability condition of Proposition 3.4 to the filtration  $0 \subset W' \subset W'^{\perp\Omega} \subset W$ .

The proof of the second statement on stability is very similar to the case of semistability, so we omit it.

Let us now consider the statement on polystability. Let  $(W, \Omega, \varphi)$  be a semistable  $U^*(2n)$ -Higgs bundle such that, for any isotropic and  $\varphi$ -invariant strict subbundle  $0 \neq$



$W' \subset W$  such that  $\deg W' = 0$ , there is another coisotropic and  $\varphi$ -invariant subbundle  $0 \neq W'' \subset W$  such that  $W = W' \oplus W''$ . We want to prove that  $(W, \Omega, \varphi)$  is polystable. Take any filtration

$$\mathcal{W} = (0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

satisfying  $W_{k-j} = W_j^{\perp\Omega}$  for any  $j$ , and the convex set  $\Lambda(\mathcal{W}, \varphi)$  defined in (3.3). Let

$$(3.8) \quad \lambda \in \Lambda(\mathcal{W}, \varphi)$$

satisfying

$$(3.9) \quad \lambda_j < \lambda_{j+1}$$

for every  $j$ , and such that

$$(3.10) \quad d(\mathcal{W}, \lambda) = 0.$$

From (3.8), (3.9) and (3.5), we conclude that

$$\mathcal{J} = \{1, \dots, k\}$$

where  $\mathcal{J}$  is given in (3.4). Therefore, every  $W_j$  in the filtration  $\mathcal{W}$  is a  $\varphi$ -invariant subbundle of  $W$ . Now, using the same arguments as in the proof of the semistability condition above, we conclude from (3.10), that

$$\deg(W_i) = 0,$$

for all  $i \in \mathcal{J}' = \{1, \dots, [k/2]\}$ . Each of these  $W_i$  is a strict isotropic and  $\varphi$ -invariant subbundle of  $W$ . In particular this holds for  $W_1$ , so from our assumption, we know that  $W/W_1$  is a  $\varphi$ -invariant coisotropic subbundle of  $W$  and  $W \cong W_1 \oplus W/W_1$ . The same is true for  $W_i$  with  $i \in \mathcal{J}'$ , hence

$$W \cong W_1 \oplus W_2/W_1 \oplus \cdots \oplus W/W_{[k/2]}.$$

For  $i \in \mathcal{J} \setminus \mathcal{J}'$ ,  $W_i$  is a strict coisotropic and  $\varphi$ -invariant subbundle of  $W$ , so  $W/W_i$  is a  $\varphi$ -invariant isotropic subbundle of  $W$ , and  $W \cong W_i \oplus W/W_i$ . Thus

$$W \cong W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_{k-1}/W_{k-2} \oplus W/W_{k-1}.$$

Since  $W_{k-j} = W_j^{\perp\Omega}$  it follows that

$$\Omega(W_i/W_{i-1}, W_j/W_{j-1}) = 0, \quad \text{unless } i + j = k + 1.$$

Moreover, since every  $W_j$  is  $\varphi$ -invariant and  $\varphi$  is symmetric with respect to  $\Omega$ , it follows that, with respect to the above decomposition of  $W$ ,

$$\varphi \in H^0 \left( \bigoplus_i \text{End}(W_i/W_{i-1}) \otimes K \right).$$

So, from Proposition 3.4,  $(W, \Omega, \varphi)$  is polystable.

The converse statement is immediate by applying the stability condition of Proposition 3.4 the filtration  $0 \subset W' \subset W'^{\perp\Omega} \subset W$  if the  $\varphi$ -invariant subbundle  $W' \subset W$  with  $\deg(W') = 0$  is isotropic, or to the filtration  $0 \subset W'^{\perp\Omega} \subset W' \subset W$  if it is coisotropic.  $\square$

In order to construct moduli spaces, we need to consider  $S$ -equivalence classes of semistable  $G$ -Higgs bundles (cf. [27]). For a stable  $G$ -Higgs bundle, its  $S$ -equivalence class coincides with its isomorphism class and for a strictly semistable  $G$ -Higgs bundle, its  $S$ -equivalence contains precisely one (up to isomorphism) representative which is polystable so this class can be thought as the isomorphism class of the unique polystable  $G$ -Higgs bundle which is  $S$ -equivalent to the given strictly semistable one.

These moduli spaces have been constructed by Schmitt in [27], using methods of Geometric Invariant Theory, showing that they carry a natural structure of complex algebraic variety.

**Definition 3.7.** *Let  $X$  be a compact Riemann of genus  $g \geq 2$ . For a reductive Lie group  $G$ , the moduli space of  $G$ -Higgs bundles over  $X$  is the complex analytic variety of isomorphism classes of polystable  $G$ -Higgs bundles. We denote it by  $\mathcal{M}_G$ :*

$$\mathcal{M}_G = \{\text{Polystable } G\text{-Higgs bundles on } X\} / \sim .$$

For a fixed topological class  $c$  of  $G$ -Higgs bundles, denote by  $\mathcal{M}_G(c)$  the moduli space of  $G$ -Higgs bundles which belong to the class  $c$ .

*Remark 3.8.* If  $G$  is an algebraic group then  $\mathcal{M}_G$  has the structure of complex algebraic variety.

**3.2. Deformation theory of  $U^*(2n)$ -Higgs bundles.** In this section, we briefly recall the deformation theory of  $G$ -Higgs bundles and, in particular, the identification of the tangent space of  $\mathcal{M}_G$  at the smooth points. All these basic notions can be found in detail in [10].

The spaces  $\mathfrak{h}^{\mathbb{C}}$  and  $\mathfrak{m}^{\mathbb{C}}$  in the Cartan decomposition of  $\mathfrak{g}^{\mathbb{C}}$  verify the relation

$$[\mathfrak{h}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}] \subset \mathfrak{m}^{\mathbb{C}}$$

hence, given  $v \in \mathfrak{m}^{\mathbb{C}}$ , there is an induced map  $\text{ad}(v)|_{\mathfrak{h}^{\mathbb{C}}} : \mathfrak{h}^{\mathbb{C}} \rightarrow \mathfrak{m}^{\mathbb{C}}$ . Applying this to a  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$ , we obtain the following complex of sheaves on the curve  $X$ :

$$C_G^\bullet(E_{H^c}, \varphi) : \mathcal{O}(E_{H^c}(\mathfrak{h}^{\mathbb{C}})) \xrightarrow{\text{ad}(\varphi)} \mathcal{O}(E_{H^c}(\mathfrak{m}^{\mathbb{C}}) \otimes K).$$

**Proposition 3.9.** *Let  $(E_{H^c}, \varphi)$  be a  $G$ -Higgs bundle over  $X$ .*

- (i) *The infinitesimal deformation space of  $(E_{H^c}, \varphi)$  is isomorphic to the first hypercohomology group  $\mathbb{H}^1(C_G^\bullet(E_{H^c}, \varphi))$  of the complex of sheaves  $C_G^\bullet(E_{H^c}, \varphi)$ ;*
- (ii) *There is a long exact sequence*

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(C_G^\bullet(E_{H^c}, \varphi)) \longrightarrow H^0(E_{H^c}(\mathfrak{h}^{\mathbb{C}})) \longrightarrow H^0(E_{H^c}(\mathfrak{m}^{\mathbb{C}}) \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^1(C_G^\bullet(E_{H^c}, \varphi)) \longrightarrow H^1(E_{H^c}(\mathfrak{h}^{\mathbb{C}})) \longrightarrow H^1(E_{H^c}(\mathfrak{m}^{\mathbb{C}}) \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^2(C_G^\bullet(E_{H^c}, \varphi)) \longrightarrow 0 \end{aligned}$$

where the maps  $H^i(E_{H^c}(\mathfrak{h}^{\mathbb{C}})) \rightarrow H^i(E_{H^c}(\mathfrak{m}^{\mathbb{C}}) \otimes K)$  are induced by  $\text{ad}(\varphi)$ .

Now, given a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$ , the complex  $C_G^\bullet(E_{H^c}, \varphi)$  defined above, becomes the complex of sheaves

$$C^\bullet(W, \Omega, \varphi) : \Lambda_\Omega^2 W \xrightarrow{\text{ad}(\varphi)} S_\Omega^2 W \otimes K$$

where  $\Lambda_\Omega^2 W$  denotes the bundle of endomorphisms of  $W$  which are skew-symmetric with respect to  $\Omega$ , and where  $\text{ad}(\varphi) = [\varphi, -]$  is given by the Lie bracket.

Proposition 3.9 applied to the case of  $U^*(2n)$ -Higgs bundles, yields the following.

**Proposition 3.10.** *Let  $(W, \Omega, \varphi)$  be a  $U^*(2n)$ -Higgs bundle over  $X$ .*

- (i) *The infinitesimal deformation space of  $(W, \Omega, \varphi)$  is isomorphic to the first hypercohomology group  $\mathbb{H}^1(C^\bullet(W, \Omega, \varphi))$  of  $C^\bullet(W, \Omega, \varphi)$ . In particular, if  $(W, \Omega, \varphi)$  represents a smooth point of  $\mathcal{M}_d$ , then*

$$T_{(W, \Omega, \varphi)} \mathcal{M} \simeq \mathbb{H}^1(C^\bullet(W, \Omega, \varphi)).$$

- (ii) *There is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathbb{H}^0(C^\bullet(W, \Omega, \varphi)) &\longrightarrow H^0(\Lambda_\Omega^2 W) \longrightarrow H^0(S_\Omega^2 W \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^1(C^\bullet(W, \Omega, \varphi)) \longrightarrow H^1(\Lambda_\Omega^2 W) \longrightarrow H^1(S_\Omega^2 W \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^2(C^\bullet(W, \Omega, \varphi)) \longrightarrow 0 \end{aligned}$$

where the maps  $H^i(\Lambda_\Omega^2 W) \rightarrow H^i(S_\Omega^2 W \otimes K)$  are induced by  $\text{ad}(\varphi) = [\varphi, -]$ .

The definition of simple  $G$ -Higgs bundle is given in [10] as follows.

**Definition 3.11.** *A  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$  is simple if  $\text{Aut}(E_{H^c}, \varphi) = \ker(\iota) \cap Z(H^c)$ , where  $Z(H^c)$  is the center of  $H^c$  and  $\iota$  is the isotropy representation (2.1).*

Contrary to the case of vector bundles, stability of a  $G$ -Higgs bundle does not imply that it is simple.

From Proposition 3.9, one has that

$$(3.11) \quad \begin{aligned} \dim \mathbb{H}^1(C_G^\bullet(E_{H^c}, \varphi)) &= \chi(E_{H^c}(\mathfrak{m}^c) \otimes K) - \chi(E_{H^c}(\mathfrak{h}^c)) + \\ &+ \dim \mathbb{H}^0(C_G^\bullet(E_{H^c}, \varphi)) + \dim \mathbb{H}^2(C_G^\bullet(E_{H^c}, \varphi)) \end{aligned}$$

where  $\chi = \dim H^0 - \dim H^1$  denotes the Euler characteristic. The summand

$$\chi(E_{H^c}(\mathfrak{m}^c) \otimes K) + \chi(E_{H^c}(\mathfrak{h}^c))$$

only depends on the topological class  $c$  of  $E_{H^c}$ , which is fixed when we consider  $\mathcal{M}_G(c)$ . In order for a polystable  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$  represent a smooth point of the moduli space  $\mathcal{M}_G$ ,  $\dim \mathbb{H}^0(C_G^\bullet(E_{H^c}, \varphi))$  and  $\dim \mathbb{H}^2(C_G^\bullet(E_{H^c}, \varphi))$  must have the minimum possible value. Indeed, we have the following Proposition 3.12 (cf. [10]), which gives sufficient conditions for a  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$  represent a smooth point of  $\mathcal{M}_G$ . It uses the construction of a  $G^c$ -Higgs bundle from a  $G$ -Higgs bundle, which we now briefly explain.

Suppose that  $G$  is a real form of  $G^c$ . The adjoint representation

$$\text{Ad}_{G^c} : G^c \rightarrow \text{Aut}(\mathfrak{g}^c)$$

of  $G^c$  on its Lie algebra restricts to  $H^c \subset G^c$  and the restriction splits as sum

$$(3.12) \quad \text{Ad}_{G^c} |_{H^c} = \text{Ad}_{H^c} \oplus \iota$$

where  $\text{Ad}_{H^c} : H^c \rightarrow \text{Aut}(\mathfrak{h}^c)$  is the adjoint representation of  $H^c$  on  $\mathfrak{h}^c$  and  $\iota : H^c \rightarrow \text{Aut}(\mathfrak{m}^c)$  is the isotropy representation (2.1). From a  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$ , we obtain a  $G^c$ -Higgs bundle as follows. Take  $E_{G^c}$  to be the holomorphic  $G^c$ -principal

bundle obtained from  $E_{H^c}$  by extending the structure group through the inclusion  $H^c \hookrightarrow G^c$ . From this construction of  $E_{G^c}$  and from (3.12), we have the splitting

$$E_{G^c} \times_{G^c} \mathfrak{g}^c = E_{H^c} \times_{H^c} \mathfrak{g}^c = E_{H^c} \times_{H^c} \mathfrak{h}^c \oplus E_{H^c} \times_{H^c} \mathfrak{m}^c.$$

So, define  $\varphi' \in H^0(E_{G^c} \times_{G^c} \mathfrak{g}^c \otimes K)$  by considering the above splitting, taking  $\varphi \in H^0(E_{H^c} \times_{H^c} \mathfrak{m}^c \otimes K)$  and taking the zero section of  $E_{H^c} \times_{H^c} \mathfrak{h}^c$ . We say that  $(E_{G^c}, \varphi')$  is the  $G^c$ -Higgs bundle associated to the  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$ . Also, when we say that we view the  $G$ -Higgs bundle  $(E_{H^c}, \varphi)$  as a  $G^c$ -Higgs bundle, it is this construction that we are referring to (see also [5]).

Now we can state the result.

**Proposition 3.12.** *Let  $(E_{H^c}, \varphi)$  be a polystable  $G$ -Higgs bundle which is stable, simple and such that  $\mathbb{H}^2(C_G^\bullet(E_{H^c}, \varphi)) = 0$ . Then it corresponds to a smooth point of the moduli space  $\mathcal{M}_G(c)$ . In particular, if  $(E_{H^c}, \varphi)$  is a simple  $G$ -Higgs bundle which is stable as a  $G^c$ -Higgs bundle, then it is a smooth point in the moduli space.*

Let  $(E_{H^c}, \varphi)$  represent a smooth point of  $\mathcal{M}_G(c)$ . The expected dimension of  $\mathcal{M}_G(c)$  is given by

$$(3.13) \quad \chi(E_{H^c}(\mathfrak{m}^c) \otimes K) - \chi(E_{H^c}(\mathfrak{h}^c)) + \dim \text{Aut}(E_{H^c}, \varphi).$$

The actual dimension of the moduli space (if non-empty) can be strictly smaller than the expected dimension. This phenomenon occurs for example in  $\mathcal{M}_{U(p,q)}$ , as explained in [1], where there is a component of dimension strictly smaller than the expected one. In fact, in that component there are no stable objects.

**3.3. Stable and non-simple  $U^*(2n)$ -Higgs bundles.** Our goal in this section is to give an explicit description of  $U^*(2n)$ -Higgs bundles which are stable but not simple.

As an example of the above construction of a  $G^c$ -Higgs bundle associated to a  $G$ -Higgs bundle, and which will be important below, consider a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$ . Then, the corresponding  $GL(2n, \mathbb{C})$ -Higgs bundle is simply  $(W, \varphi)$ . So we forget the symplectic form on the vector bundle  $W$ .

**Proposition 3.13.** *Let  $(W, \Omega, \varphi)$  be a  $U^*(2n)$ -Higgs bundle and  $(W, \varphi)$  be the corresponding  $GL(2n, \mathbb{C})$ -Higgs bundle. Then  $(W, \Omega, \varphi)$  is semistable if and only if  $(W, \varphi)$  is semistable.*

*Proof.* If  $(W, \varphi)$  is semistable, then it is obvious, taking into account Propositions 3.2 and 3.6, that  $(W, \Omega, \varphi)$  is semistable.

Suppose then that  $(W, \Omega, \varphi)$  is semistable. Let  $W' \subset W$  be a  $\varphi$ -invariant subbundle of  $W$ . Since  $W'^{\perp\Omega}$  is the subbundle of  $W$  defined as the kernel of the projection  $W \rightarrow W'^*$  given by  $v \mapsto \Omega(v, -)$ , and since  $\deg(W) = 0$ , we have

$$(3.14) \quad \deg(W'^{\perp\Omega}) = \deg(W').$$

The fact that  $\varphi$  is symmetric with respect to  $\Omega$ , i.e. (2.2) holds, implies that  $W'^{\perp\Omega}$  is also  $\varphi$ -invariant.

Consider the exact sequence

$$(3.15) \quad 0 \longrightarrow N \longrightarrow W' \oplus W'^{\perp\Omega} \longrightarrow M \longrightarrow 0,$$

where  $N$  and  $M$  are the saturations of the sheaves  $W' \cap W'^{\perp\Omega}$  and  $W' + W'^{\perp\Omega}$  respectively. We have that  $M = N^{\perp\Omega}$ , so

$$0 \longrightarrow M \longrightarrow W \longrightarrow N^* \longrightarrow 0$$

and thus  $\deg(M) = \deg(N)$ . It follows from (3.14) and (3.15) that  $\deg(W') = \deg(N)$ . But,  $N$  is clearly  $\varphi$ -invariant and also isotropic, so from the semistability of  $(W, \Omega, \varphi)$ ,  $\deg(N) \leq 0$  i.e.  $\deg(W') \leq 0$ . Hence, from Proposition 3.2,  $(W, \varphi)$  is semistable.  $\square$

**Proposition 3.14.** *Let  $(W, \Omega, \varphi)$  be a  $U^*(2n)$ -Higgs bundle. Then  $(W, \Omega, \varphi)$  is stable if and only if*

$$(W, \Omega, \varphi) = \bigoplus (W_i, \Omega_i, \varphi_i)$$

where  $(W_i, \Omega_i, \varphi_i)$  are  $U^*(\text{rk}(W_i))$ -Higgs bundles such that the  $\text{GL}(\text{rk}(W_i), \mathbb{C})$ -Higgs bundles  $(W_i, \varphi_i)$  are stable and nonisomorphic.

*Proof.* Suppose that  $(W, \Omega, \varphi)$  is stable. From Proposition 3.13, follows that  $(W, \varphi)$  is semistable. If it is stable, then there is nothing to prove. So, assume that  $(W, \varphi)$  is strictly semistable, and let  $W' \subset W$  be a  $\varphi$ -invariant subbundle of  $W$  of degree 0. The stability of  $(W, \Omega, \varphi)$  says that  $W'$  is not isotropic. As in the proof of the previous proposition, consider the exact sequence

$$(3.16) \quad 0 \longrightarrow N \longrightarrow W' \oplus W'^{\perp\Omega} \longrightarrow M \longrightarrow 0,$$

where  $N$  and  $M$  are the saturations of the sheaves  $W' \cap W'^{\perp\Omega}$  and  $W' + W'^{\perp\Omega}$  respectively. From the sequence

$$0 \longrightarrow W'^{\perp\Omega} \longrightarrow W \longrightarrow W'^* \longrightarrow 0$$

we have  $\deg(W'^{\perp\Omega}) = \deg(W') = 0$  so, from (3.16),

$$\deg(N) + \deg(M) = 0.$$

Recall again that, since  $W'$  is  $\varphi$ -invariant, then  $W'^{\perp\Omega}$  is also  $\varphi$ -invariant, so  $N \subset W$  is  $\varphi$ -invariant as well and, since it is isotropic, we must have  $\deg(N) < 0$ , if  $N \neq 0$ . But, if this occurs, we have  $\deg(M) > 0$  contradicting (since  $M$  is  $\varphi$ -invariant) the semistability of  $(W, \varphi)$ . We must therefore have  $N = 0$ , hence

$$(W, \varphi) = (W', \varphi|_{W'}) \oplus (W'^{\perp\Omega}, \varphi|_{W'^{\perp\Omega}}).$$

Now,  $W' \not\cong W'^{\perp\Omega}$ . In fact, if  $W' \cong W'^{\perp\Omega}$  then the inclusion  $W' \subset W' \oplus W' = W$  given by  $w \mapsto (w, \sqrt{-1}w)$  gives rise to a degree 0 isotropic,  $\varphi$ -invariant subbundle of  $W$ , contradicting the stability of  $(W, \Omega, \varphi)$ . Finally, notice that we must have

$$\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

with respect to the decomposition  $W = W' \oplus W'^{\perp\Omega}$ , where  $\omega_1 : W' \rightarrow W'^*$  and  $\omega_2 : W'^{\perp\Omega} \rightarrow (W'^{\perp\Omega})^*$  are skew-symmetric isomorphisms. The symplectic form  $\Omega$  therefore splits into a sum of symplectic forms  $\Omega_1 \oplus \Omega_2$ , and we have a splitting

$$(W, \Omega, \varphi) = (W', \Omega_1, \varphi_1) \oplus (W'^{\perp\Omega}, \Omega_2, \varphi_2).$$

Now, if  $(W', \varphi_1)$  is stable as a  $\text{GL}(\text{rk}(W'), \mathbb{C})$ -Higgs bundle and if the same happens to  $(W'^{\perp\Omega}, \varphi_2)$ , then we are done. If not, then we repeat the argument and, by induction on the rank of  $W$ , we see that  $(W, \Omega, \varphi)$  has the desired form.

To prove the converse, suppose that

$$(W, \Omega, \varphi) = \bigoplus (W_i, \Omega_i, \varphi_i)$$

as stated and let  $W' \subset W$  be a  $\varphi$ -invariant subbundle of degree 0. Since each  $(W_i, \varphi_i)$  is a stable  $\mathrm{GL}(\mathrm{rk}(W_i), \mathbb{C})$ -Higgs bundle, then the projection  $W' \rightarrow W_i$  must be either zero or surjective. Thus,  $(W', \varphi|_{W'})$  is a direct sum of some of the  $(W_i, \varphi_i)$ , so  $W'$  is not isotropic and therefore  $(W, \Omega, \varphi)$  is stable.  $\square$

Applying Definition 3.11 to the case  $G = \mathrm{U}^*(2n)$ , we have:

**Lemma 3.15.** *A  $\mathrm{U}^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  is simple if and only if  $\mathrm{Aut}(W, \Omega, \varphi) = \mathbb{Z}/2$ .*

**Corollary 3.16.** *Let  $(W, \Omega, \varphi)$  be a stable  $\mathrm{U}^*(2n)$ -Higgs bundle. Then  $(W, \Omega, \varphi)$  is simple if and only if the  $\mathrm{GL}(2n, \mathbb{C})$ -Higgs bundle  $(W, \varphi)$  is stable.*

*Proof.* Since  $(W, \Omega, \varphi)$  is stable, we have, from Proposition 3.14,

$$(3.17) \quad (W, \Omega, \varphi) = \bigoplus_{i=1}^r (W_i, \Omega_i, \varphi_i),$$

so

$$(W, \varphi) = \bigoplus_{i=1}^r (W_i, \varphi_i)$$

where  $(W_i, \varphi_i)$  are stable Higgs bundles. Since stable Higgs bundles are simple (cf. [17]) then  $\mathrm{Aut}(W_i, \varphi_i) = \mathbb{C}^*$ . This means that, for each  $i$ ,  $\mathrm{Aut}(W_i, \Omega_i, \varphi_i) = \mathbb{Z}/2$ , because the automorphisms must preserve the symplectic form  $\Omega_i$ . From (3.17), we have therefore

$$\mathrm{Aut}(W, \Omega, \varphi) = (\mathbb{Z}/2)^r.$$

It follows that  $(W, \Omega, \varphi)$  is simple if and only if  $r = 1$  i.e.  $(W, \varphi)$  is a stable Higgs bundle.  $\square$

Now the description of stable and non-simple  $\mathrm{U}^*(2n)$ -Higgs bundles is immediately obtained.

**Proposition 3.17.** *A  $\mathrm{U}^*(2n)$ -Higgs bundle is stable and non-simple if and only if it decomposes as a direct sum of stable and simple  $\mathrm{U}^*(2n)$ -Higgs bundles. In other words, a  $\mathrm{U}^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  is stable and non-simple if and only if*

$$(W, \Omega, \varphi) = \bigoplus_{i=1}^r (W_i, \Omega_i, \varphi_i)$$

where  $(W_i, \Omega_i, \varphi_i)$  are stable and simple  $\mathrm{U}^*(\mathrm{rk}(W_i))$ -Higgs bundles and  $r > 1$ .

The following result will be important below. It is straightforward from Proposition 3.12, from the fact that the complexification of  $\mathrm{U}^*(2n)$  is  $\mathrm{GL}(2n, \mathbb{C})$  and from Corollary 3.16:

Let  $\mathcal{M}_{\mathrm{U}^*(2n)}$  denote the moduli space of polystable  $\mathrm{U}^*(2n)$ -Higgs bundles.

**Proposition 3.18.** *A stable and simple  $\mathrm{U}^*(2n)$ -Higgs bundle corresponds to a smooth point of the moduli space  $\mathcal{M}_{\mathrm{U}^*(2n)}$ .*

So, from [27], at a point of  $\mathcal{M}_{U^*(2n)}$  represented by a stable and simple object, there exists a local universal family, hence the dimension of the component of  $\mathcal{M}_{U^*(2n)}$  containing that point is the expected dimension given by (3.13), which, for  $G = U^*(2n)$  is easily seen to be equal to

$$4n^2(g-1).$$

**3.4. Polystable  $U^*(2n)$ -Higgs bundles.** Now we look at polystable  $U^*(2n)$ -Higgs bundles. First notice that we can realize  $GL(n, \mathbb{C})$  as a subgroup of  $U^*(2n)$ , using the injection

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix}.$$

When restricted to the unitary group  $U(n) \subset GL(n, \mathbb{C})$  we obtain the injection

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}.$$

**Theorem 3.19.** *Let  $(W, \Omega, \varphi)$  be a polystable  $U^*(2n)$ -Higgs bundle. There is a decomposition of  $(W, \Omega, \varphi)$  as a sum of stable  $G_i$ -Higgs bundles, where  $G_i$  is one of the following subgroups of  $U^*(2n)$ :  $U^*(2n_i)$ ,  $GL(n_i, \mathbb{C})$ ,  $Sp(2n_i)$  or  $U(n_i)$  ( $n_i \leq n$ ).*

*Proof.* Since  $(W, \Omega, \varphi)$  is polystable, we know, from Proposition 3.4, that for every filtration

$$\mathcal{W} = (0 = W_0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W)$$

such that  $W_{k-j} = W_j^{\perp \Omega}$ , and any

$$\lambda \in \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{R}^k \mid \lambda_i < \lambda_{i+1} \text{ and } \lambda_{k-i+1} = -\lambda_i \text{ for any } i\},$$

such that  $\varphi \in H^0(N(\mathcal{W}, \lambda))$  and  $d(\mathcal{W}, \lambda) = 0$ , there is an isomorphism

$$(3.18) \quad W \simeq W_1 \oplus W_2/W_1 \oplus \cdots \oplus W_k/W_{k-1}$$

such that

$$(3.19) \quad \Omega(W_i/W_{i-1}, W_j/W_{j-1}) = 0, \quad \text{unless } i+j = k+1$$

and that, via this isomorphism,

$$(3.20) \quad \varphi \in H^0\left(\bigoplus_i \text{End}(W_i/W_{i-1}) \otimes K\right).$$

Now we analyze the possible cases. Conditions (3.19) and (3.20) tell us that, with respect to decomposition (3.18), we have

$$(3.21) \quad \omega = \begin{pmatrix} 0 & 0 & 0 & \cdots & -\omega_1^t \\ \vdots & \cdots & 0 & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ \omega_1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where  $\omega_i : W_i/W_{i-1} \xrightarrow{\cong} (W_{k+1-i}/W_{k-i})^*$  is the isomorphism induced by  $\Omega$ , and that

$$\varphi(W_i/W_{i-1}) \subset W_i/W_{i-1} \otimes K,$$

for all  $i = 1, \dots, k$ , so we write

$$\varphi_i = \varphi|_{W_i/W_{i-1}}.$$

Hence, if  $i \neq \frac{k+1}{2}$ , from (3.19), the symplectic form  $\Omega$  does not restricts to a symplectic form on  $W_i/W_{i-1}$ , and we deduce that

$$(W_i/W_{i-1}, \varphi_i)$$

is a  $\mathrm{GL}(\mathrm{rk}(W_i/W_{i-1}), \mathbb{C})$ -Higgs bundle, being a  $\mathrm{U}(\mathrm{rk}(W_i/W_{i-1}))$ -Higgs bundle if and only if  $\varphi_i = 0$ .

On the other hand, the symplectic form  $\Omega$  restricts to a symplectic form  $\Omega_{\frac{k+1}{2}}$  on  $W_{\frac{k+1}{2}}/W_{\frac{k-1}{2}}$ , and we deduce that

$$(W_{\frac{k+1}{2}}/W_{\frac{k-1}{2}}, \Omega_{\frac{k+1}{2}}, \varphi_{\frac{k+1}{2}})$$

is a  $\mathrm{U}^*(\mathrm{rk}(W_{\frac{k+1}{2}}/W_{\frac{k-1}{2}}))$ -Higgs bundle, being a  $\mathrm{Sp}(\mathrm{rk}(W_{\frac{k+1}{2}}/W_{\frac{k-1}{2}}))$ -Higgs bundle if and only if  $\varphi_{\frac{k+1}{2}} = 0$ . Of course, this case can only occur if  $k$  is odd.

Each summand in this decomposition is also polystable (one way of seeing this is by using the Hitchin-Kobayashi correspondence between polystable  $G$ -Higgs bundles and solutions to the Hitchin equations; cf. [10]). Hence, for each summand which is a  $\mathrm{Sp}(2n_i)$ - or  $\mathrm{GL}(n_i, \mathbb{C})$ - or  $\mathrm{U}(n_i)$ -Higgs bundle we know that we can continue the process for these groups, until we obtain a decomposition where all summands are stable  $\mathrm{Sp}(2n_i)$ - or  $\mathrm{GL}(n_i, \mathbb{C})$ - or  $\mathrm{U}(n_i)$ -Higgs bundles: for  $\mathrm{U}(n_i)$ -Higgs bundles (i.e. holomorphic vector bundles) this is proved in [28]; the proof for the case of  $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles can be found in [22] and for  $\mathrm{Sp}(2n)$ -Higgs bundles (i.e. symplectic vector bundles) this is proved in [19] (see also [23]). On the other hand, for  $\mathrm{U}^*(2n_i)$ -Higgs bundle we simply iterate the above process. Finally we obtain a decomposition where all summands are stable  $G_i$ -Higgs bundles.  $\square$

#### 4. THE HITCHIN PROPER FUNCTIONAL AND THE MINIMA SUBVARIETIES

Here we use the method introduced by Hitchin in [17] to study the topology of moduli space  $\mathcal{M}_G$  of  $G$ -Higgs bundles.

Define

$$f : \mathcal{M}_G(c) \longrightarrow \mathbb{R}$$

by

$$(4.1) \quad f(E_{H^c}, \varphi) = \|\varphi\|_{L^2}^2 = \int_X |\varphi|^2 \mathrm{dvol}.$$

This function  $f$  is usually called the *Hitchin functional*.

Here we are using the *harmonic metric* (cf. [7, 8]) on  $E_{H^c}$  to define  $\|\varphi\|_{L^2}$ . So we are using the identification between  $\mathcal{M}_G(c)$  with the space of gauge-equivalent solutions to Hitchin's equations. We opt to work with  $\mathcal{M}_G(c)$ , because in this case we have more algebraic tools at our disposal. We shall make use of the tangent space of  $\mathcal{M}_G(c)$ , and



we know from [17] that the above identification induces a diffeomorphism between the corresponding tangent spaces.

Hitchin proved in [17, 18] that the function  $f$  is proper and therefore it attains a minimum on each closed subspace of  $\mathcal{M}_G = \bigcup_c \mathcal{M}_G(c)$ . Moreover, we have the following result from general topology.

**Proposition 4.1.** *Let  $\mathcal{M}' \subseteq \mathcal{M}_G$  be a closed subspace and let  $\mathcal{N}' \subset \mathcal{M}'$  be the subspace of local minima of  $f$  on  $\mathcal{M}'$ . If  $\mathcal{N}'$  is connected then so is  $\mathcal{M}'$ .*

In our case, the Hitchin functional

$$f : \mathcal{M}_{U^*(2n)} \longrightarrow \mathbb{R}$$

is given by

$$(4.2) \quad f(W, \Omega, \varphi) = \|\varphi\|_{L^2}^2 = \frac{\sqrt{-1}}{2} \int_X \text{tr}(\varphi \wedge \varphi^*) \text{dvol}.$$

Recall from Proposition 3.18 which guarantees that a stable and simple  $U^*(2n)$ -Higgs bundle represents a smooth point on  $\mathcal{M}_{U^*(2n)}$ .

Away from the singular locus of  $\mathcal{M}_{U^*(2n)}$ , the Hitchin functional  $f$  is a moment map for the Hamiltonian  $S^1$ -action on  $\mathcal{M}_{U^*(2n)}$  given by

$$(4.3) \quad (V, \varphi) \mapsto (V, e^{\sqrt{-1}\theta} \varphi).$$

From this it follows immediately that a smooth point of  $\mathcal{M}_{U^*(2n)}$  is a critical point of  $f$  if and only if it is a fixed point of the  $S^1$ -action. Let us then study the fixed point set of the given action (this is analogous to [18] and [2]).

Let  $(W, \Omega, \varphi)$  represent a stable and simple (hence smooth) fixed point. Then either  $\varphi = 0$  or (since the action is on  $\mathcal{M}_{U^*(2n)}$ ) there is a one-parameter family of gauge transformations  $g(\theta)$  such that  $g(\theta) \cdot (W, \Omega, \varphi) = (W, \Omega, e^{\sqrt{-1}\theta} \varphi)$ .

In the latter case, let

$$(4.4) \quad \psi = \frac{d}{d\theta} g(\theta)|_{\theta=0}$$

be the infinitesimal gauge transformation generating this family.  $(W, \Omega, \varphi)$  is then what is called a *complex variation of Hodge structure* or a *Hodge bundle* (cf. [17, 18, 30]). This means that

$$(W, \varphi) = \left( \bigoplus F_j, \sum \varphi_j \right)$$

where the  $F_j$ 's are the eigenbundles of the infinitesimal gauge transformation  $\psi$ : over  $F_j$ ,

$$(4.5) \quad \psi = \sqrt{-1}j \in \mathbb{C},$$

and where  $\varphi_j = \varphi|_{F_j}$  is a map

$$(4.6) \quad \varphi_j : F_j \longrightarrow F_{j+1} \otimes K.$$

Since  $g(\theta)$  is an automorphism of  $(W, \Omega)$ , it follows from (4.4) that  $\psi$  is skew-symmetric with respect to  $\Omega$ . Thus, using (4.5) we have that, if  $v_j \in F_j$  and  $v_i \in F_i$ ,

$$\sqrt{-1}j\Omega(v_j, v_i) = \Omega(\psi v_j, v_i) = -\Omega(v_j, \psi v_i) = -\sqrt{-1}i\Omega(v_j, v_i).$$

Then  $F_j$  and  $F_i$  are therefore orthogonal under  $\Omega$  unless  $i + j = 0$ , and therefore  $\omega : W \rightarrow W^*$  yields an isomorphism

$$(4.7) \quad \omega_j = \omega|_{F_j} : F_j \xrightarrow{\cong} F_{-j}^*.$$

This means that

$$(4.8) \quad W = F_{-m} \oplus \cdots \oplus F_m$$

for some  $m \geq 1/2$  integer or half-integer.

Using these isomorphisms and (2.2), we see that

$$(\varphi_{-j-1}^t \otimes 1_K)\omega_j = (\omega_{j+1} \otimes 1_K)\varphi_j$$

for  $j \in \{-m, \dots, m\}$ .

The Cartan decomposition of  $\mathfrak{g}^{\mathbb{C}}$  induces a decomposition of vector bundles

$$E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) = E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}) \oplus E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$$

where  $E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}})$  (resp.  $E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}})$ ) is the adjoint bundle, associated to the adjoint representation of  $H^{\mathbb{C}}$  on  $\mathfrak{g}^{\mathbb{C}}$  (resp.  $\mathfrak{h}^{\mathbb{C}}$ ). For the group  $U^*(2n)$ , we have  $E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) = \text{End}(W)$  and we already know that  $E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}) = \Lambda_{\Omega}^2 W$  and  $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = S_{\Omega}^2 W$ . The involution in  $\text{End}(W)$  defining the above decomposition is  $\theta : \text{End}(W) \rightarrow \text{End}(W)$  defined by

$$(4.9) \quad \theta(A) = -(\omega A \omega^{-1})^t.$$

Its  $+1$ -eigenbundle is  $\Lambda_{\Omega}^2 W$  and its  $-1$ -eigenbundle is  $S_{\Omega}^2 W$ .

We also have a decomposition of this vector bundle as

$$(4.10) \quad \text{End}(W) = \bigoplus_{k=-2m}^{2m} U_k$$

where

$$U_k = \bigoplus_{i-j=k} \text{Hom}(F_j, F_i).$$

From (4.5), this is the  $\sqrt{-1}k$ -eigenbundle for the adjoint action  $\text{ad}(\psi) : \text{End}(W) \rightarrow \text{End}(W)$  of  $\psi$ . We say that  $U_k$  is the subspace of  $\text{End}(W)$  with *weight*  $k$ .

Write

$$U_{i,j} = \text{Hom}(F_j, F_i).$$

The restriction of the involution  $\theta$ , defined in (4.9), to  $U_{i,j}$  gives an isomorphism

$$(4.11) \quad \theta : U_{i,j} \xrightarrow{\cong} U_{-j,-i}$$

so  $\theta$  restricts to

$$\theta : U_k \longrightarrow U_k.$$

Write

$$U^+ = \Lambda_{\Omega}^2 W \quad \text{and} \quad U^- = S_{\Omega}^2 W$$

so that  $E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}) = U^+$  and  $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = U^-$ . Let also

$$U_k^+ = U_k \cap U^+$$

and

$$U_k^- = U_k \cap U^-$$

so that  $U_k = U_k^+ \oplus U_k^-$  is the corresponding eigenbundle decomposition. Hence

$$U^+ = \bigoplus_k U_k^+$$

and

$$U^- = \bigoplus_k U_k^-.$$

Observe that  $\varphi \in H^0(U_1^- \otimes K)$ .

The map  $\text{ad}(\varphi) = [\varphi, -]$  interchanges  $U^+$  with  $U^-$  and therefore maps  $U_k^\pm$  to  $U_{k+1}^\mp \otimes K$ . So, for each  $k$ , we have a weight  $k$  subcomplex of the complex  $C^\bullet(W, \Omega, \varphi)$  defined in Proposition 3.10:

$$(4.12) \quad C_k^\bullet(W, \Omega, \varphi) : U_k^+ \xrightarrow{\text{ad}(\varphi)} U_{k+1}^- \otimes K.$$

From Propositions 3.10 and 3.18, if a  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  is stable and simple, its infinitesimal deformation space is

$$\mathbb{H}^1(C^\bullet(W, \Omega, \varphi)) = \bigoplus_k \mathbb{H}^1(C_k^\bullet(W, \Omega, \varphi)).$$

We say that  $\mathbb{H}^1(C_k^\bullet(W, \Omega, \varphi))$  is the subspace of  $\mathbb{H}^1(C^\bullet(W, \Omega, \varphi))$  with *weight*  $k$ .

By Hitchin's computations in [18] (see also [12]), we have the following result which gives us a way to compute the eigenvalues of the Hessian of the Hitchin functional  $f$  at a smooth critical point.

**Proposition 4.2.** *Let  $(W, \Omega, \varphi)$  be a smooth  $U^*(2n)$ -Higgs bundle which represents a critical point of the Hitchin function  $f$ . The eigenspace of the Hessian of  $f$  corresponding to the eigenvalue  $k$  is*

$$\mathbb{H}^1(C_{-k}^\bullet(W, \Omega, \varphi)).$$

*In particular,  $(W, \Omega, \varphi)$  is a local minimum of  $f$  if and only if  $\mathbb{H}^1(C^\bullet(W, \Omega, \varphi))$  has no subspaces with positive weight.*

For each  $k$ , consider the complex (4.12) and let

$$\chi(C_k^\bullet(W, \Omega, \varphi)) = \dim \mathbb{H}^0(C_k^\bullet(W, \Omega, \varphi)) - \dim \mathbb{H}^1(C_k^\bullet(W, \Omega, \varphi)) + \dim \mathbb{H}^2(C_k^\bullet(W, \Omega, \varphi)).$$

**Lemma 4.3.** *Let  $(W, \Omega, \varphi)$  be a stable  $U^*(2n)$ -Higgs bundle which corresponds to a critical point of  $f$ . Then  $\chi(C_k^\bullet(W, \Omega, \varphi)) \leq (g-1)(2 \text{rk}(\text{ad}(\varphi)|_{U_k^+}) - \text{rk}(U_k^+) - \text{rk}(U_{k+1}^-))$ . Furthermore,  $\chi(C_k^\bullet(W, \Omega, \varphi)) = 0$  if and only if  $\text{ad}(\varphi)|_{U_k^+} : U_k^+ \rightarrow U_{k+1}^- \otimes K$  is an isomorphism.*

*Proof.* This is essentially Lemma 3.11 of [4] (see also Proposition 4.4 of [1]). The proof in those papers is for  $\text{GL}(n, \mathbb{C})$  and  $U(p, q)$ -Higgs bundles, but the argument works in the general setting of  $G$ -Higgs bundles (see Remark 4.16 of [1]): the key facts are that for a stable  $G$ -Higgs bundle,  $(E_{H^c}, \varphi)$ , the Higgs vector bundle  $(E_{H^c} \times_{\text{Ad}} \mathfrak{g}^c, \text{ad}(\varphi))$  is semistable, and that there is a natural  $\text{ad}$ -invariant isomorphism  $E_{H^c} \times_{\text{Ad}} \mathfrak{g}^c \cong (E_{H^c} \times_{\text{Ad}} \mathfrak{g}^c)^*$  given by an invariant pairing on  $\mathfrak{g}^c$  (e.g. the Killing form). So we will only give a sketch of the proof here.

In the following we shall use the abbreviated notations  $C_k^\bullet = C_k^\bullet(W, \Omega, \varphi)$  and

$$\varphi_k^\pm = \text{ad}(\varphi)|_{U_k^\pm} : U_k^+ \longrightarrow U_{k+1}^- \otimes K.$$

By the Riemann-Roch theorem we have

$$(4.13) \quad \chi(C_k^\bullet) = (1 - g)(\text{rk}(U_k^+) + \text{rk}(U_{k+1}^-)) + \text{deg}(U_k^+) - \text{deg}(U_{k+1}^-),$$

thus we can prove the inequality stated in the lemma by estimating the difference  $\text{deg}(U_k^+) - \text{deg}(U_{k+1}^-)$ . In order to do this, we note first that there are short exact sequences of sheaves

$$0 \rightarrow \ker(\varphi_k^+) \rightarrow U_k^+ \rightarrow \text{im}(\varphi_k^+) \rightarrow 0$$

and

$$0 \rightarrow \text{im}(\varphi_k^+) \rightarrow U_{k+1}^- \otimes K \rightarrow \text{coker}(\varphi_k^+) \rightarrow 0.$$

It follows that

$$(4.14) \quad \text{deg}(U_k^+) - \text{deg}(U_{k+1}^-) = \text{deg}(\ker(\varphi_k^+)) + (2g - 2)\text{rk}(U_{k+1}^-) - \text{deg}(\text{coker}(\varphi_k^+)).$$

The following inequalities are proved in the proof of Lemma 3.11 in [4]:

$$(4.15) \quad \text{deg}(\ker(\varphi_k^+)) \leq 0,$$

$$(4.16) \quad -\text{deg}(\text{coker}(\varphi_k^+)) \leq (2g - 2)(-\text{rk}(U_{k+1}^-) + \text{rk}(\varphi_k^+)).$$

Combining (4.15) and (4.16) with (4.14) we obtain

$$\text{deg}(U_k^+) - \text{deg}(U_{k+1}^-) \leq (2g - 2)\text{rk}(\varphi_k^+),$$

which, together with (4.13), proves the inequality stated in the lemma.

Finally, if  $\chi(C_k^\bullet) = 0$  then

$$\text{rk}(\varphi_k^+) = \text{rk}(U_k^+) = \text{rk}(U_{k+1}^- \otimes K)$$

hence  $\text{deg}(\ker(\varphi_k^+)) = 0$ . Moreover, it is shown again in the proof of Lemma 3.11 [4] that  $\text{deg}(\text{coker}(\varphi_k^+)) = 0$ . Thus, from (4.14),

$$\text{deg}(U_k^+) = \text{deg}(U_{k+1}^- \otimes K),$$

showing that  $\varphi_k^+$  is an isomorphism.  $\square$

The following result is fundamental for the description of the stable and simple local minima of  $f$ .

**Theorem 4.4.** *Let  $(W, \Omega, \varphi) \in \mathcal{M}_{U^*(2n)}$  be a stable and simple critical point of the Hitchin functional  $f$ . Then  $(W, \Omega, \varphi)$  is a local minimum if and only if either  $\varphi = 0$  or*

$$\text{ad}(\varphi)|_{U_k^+} : U_k^+ \longrightarrow U_{k+1}^- \otimes K$$

*is an isomorphism for all  $k \geq 1$ .*

*Proof.* Suppose  $\varphi \neq 0$  and that  $\text{ad}(\varphi)|_{U_k^+}$  is an isomorphism for every  $k \geq 1$ . Then, Lemma 4.3 says that this is equivalent to

$$\dim \mathbb{H}^1(C_k^\bullet(W, \Omega, \varphi)) = \dim \mathbb{H}^0(C_k^\bullet(W, \Omega, \varphi)) + \dim \mathbb{H}^2(C_k^\bullet(W, \Omega, \varphi))$$

for all  $k \geq 1$ . Now, since  $(W, \Omega, \varphi)$  is stable and simple, then it is stable as a  $\text{GL}(2n, \mathbb{C})$ -Higgs bundle, by Corollary 3.16. Furthermore,  $U^*(2n)$  is semisimple, so

from Proposition 3.17 of [10] follows that  $\mathbb{H}^0(C^\bullet(W, \Omega, \varphi)) = \mathbb{H}^2(C^\bullet(W, \Omega, \varphi)) = 0$ , so  $\mathbb{H}^0(C_k^\bullet(W, \Omega, \varphi)) = \mathbb{H}^2(C_k^\bullet(W, \Omega, \varphi)) = 0$  for every  $k \geq 1$ . Then  $\mathbb{H}^1(C_k^\bullet(W, \Omega, \varphi)) = 0$  for every  $k \geq 1$  and the result follows from Proposition 4.2.

The converse statement is now immediate.  $\square$

Using this, one can describe the smooth local minima of the Hitchin functional  $f$ .

**Proposition 4.5.** *Let the  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  be a critical point of the Hitchin functional  $f$  such that  $(W, \Omega, \varphi)$  is stable and simple (hence smooth). Then  $(W, \Omega, \varphi)$  represents a local minimum if and only if  $\varphi = 0$ .*

*Proof.* Suppose that  $(W, \Omega, \varphi)$  is a critical point of  $f$  with  $\varphi \neq 0$ . Hence, as explained above, we have the decompositions (4.8) and (4.10) of  $W$  and of  $\text{End}(W)$  respectively.

Consider

$$\text{ad}(\varphi)|_{U_{2m}^+} : U_{2m}^+ \longrightarrow U_{2m+1}^- \otimes K.$$

We have that  $U_{2m+1}^- \otimes K = 0$ , but  $U_{2m}^+ \neq 0$ . Indeed, if  $U_{2m}^+ = 0$ , then

$$\text{Hom}(F_{-m}, F_m) = U_{2m} = U_{2m}^-$$

i.e. given any  $g : F_{-m} \rightarrow F_m$ , we would have  $g \in S_\Omega^2 W$ , thus

$$\omega_m g = g^t \omega_{-m} = -(\omega_m g)^t$$

where  $\omega_{\pm m}$  are the isomorphisms defined in (4.7). In other words,  $\omega_m g \in H^0(\Lambda^2 F_{-m}^*)$ , for any  $g$ . But  $\omega_m$  is an isomorphism, so any map  $F_{-m} \rightarrow F_{-m}^*$  is of the form  $\omega_m g$ , for some  $g$ . This shows that  $H^0(\text{Hom}(F_{-m}, F_{-m}^*)) = H^0(\Lambda^2 F_{-m}^*)$  which is clearly not possible.

So,  $U_{2m}^+ \neq 0$ , therefore  $\text{ad}(\varphi)|_{U_{2m}^+}$  is not an isomorphism and by the previous theorem,  $(W, \Omega, \varphi)$  is not a local minimum of the Hitchin functional.  $\square$

In [18], Hitchin observed that the Hitchin functional is additive with respect to direct sum of Higgs bundles. In our case this means that  $f(\bigoplus(V_i, \Omega_i, \varphi_i)) = \sum f(V_i, \Omega_i, \varphi_i)$ .

**Proposition 4.6.** *A stable  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  represents a local minimum of  $f$  if and only if  $\varphi = 0$ .*

*Proof.* If  $(W, \Omega, \varphi)$  is simple, then this is true from Proposition 4.5. So, assume that the local minimum  $(W, \Omega, \varphi)$  of  $f$  is stable and non-simple. Then, from Proposition 3.17, we know that  $(W, \Omega, \varphi)$  decomposes as a direct sum of stable and simple  $U^*(2n_i)$ -Higgs bundles on the corresponding lower rank moduli spaces. Moreover, using the additivity of  $f$ , we know that these are also local minima of  $f$ . So, in those moduli spaces we can apply Proposition 4.5, and the additivity of  $f$  implies that the result follows.  $\square$

Now we can give the description of the subvariety of local minima of the Hitchin functional  $f$ .

**Theorem 4.7.** *A polystable  $U^*(2n)$ -Higgs bundle  $(W, \Omega, \varphi)$  represents a local minimum if and only if  $\varphi = 0$ .*

*Proof.* From Theorem 3.19 we know that a polystable minima of  $f$  decomposes as a direct sum of stable  $G_i$ -Higgs bundles where  $G_i = \mathrm{U}^*(2n_i)$ ,  $\mathrm{Sp}(2n_i)$ ,  $\mathrm{GL}(n_i, \mathbb{C})$  or  $\mathrm{U}(n_i)$ . Now, for the groups  $\mathrm{Sp}(n_i)$  or  $\mathrm{U}(n_i)$  it is clear that the local minima of  $f$  on the corresponding lower rank moduli spaces must have zero Higgs field (these groups are compact). For  $\mathrm{GL}(n_i, \mathbb{C})$  it is well-known (cf. [17]) that stable local minima of  $f$  on the corresponding lower rank moduli space must also have  $\varphi_i = 0$ . For stable  $\mathrm{U}^*(2n_i)$ -Higgs bundle, we can apply Proposition 4.6 to draw the same conclusion, and the result is proved.  $\square$

## 5. CONNECTED COMPONENTS OF THE SPACE OF $\mathrm{U}^*(2n)$ -HIGGS BUNDLES

From Theorem 4.7 we conclude that the subvariety  $\mathcal{N}_{\mathrm{U}^*(2n)}$  of local minima of the Hitchin functional  $f : \mathcal{M}_{\mathrm{U}^*(2n)} \rightarrow \mathbb{R}$  is the moduli space of  $\mathrm{Sp}(2n, \mathbb{C})$ -principal bundles or, in the language of Higgs bundles, is the moduli space of  $\mathrm{Sp}(2n)$ -Higgs bundles:

$$\mathcal{N}_{\mathrm{U}^*(2n)} \cong \mathcal{M}_{\mathrm{Sp}(2n)}.$$

Ramanathan has shown [25, 26] that if  $G$  is a connected reductive group then there is a bijective correspondence between  $\pi_0$  of the moduli space of  $G$ -principal bundles and  $\pi_1 G$ . Hence, since  $\mathrm{Sp}(2n)$  is simply-connected, it follows that  $\mathcal{M}_{\mathrm{Sp}(2n)}$  is connected and, therefore, the same is true for  $\mathcal{N}_{\mathrm{U}^*(2n)}$ . So, using Proposition 4.1, we can state our result.

**Theorem 5.1.** *Let  $X$  be a compact Riemann surface of genus  $g \geq 2$  and let  $\mathcal{M}_{\mathrm{U}^*(2n)}$  be the moduli space of  $\mathrm{U}^*(2n)$ -Higgs bundles. Then  $\mathcal{M}_{\mathrm{U}^*(2n)}$  is connected.*

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