# TOPOLOGICAL MIRROR SYMMETRY FOR PARABOLIC HIGGS BUNDLES 

PETER B. GOTHEN AND ANDRÉ G. OLIVEIRA


#### Abstract

We prove the topological mirror symmetry conjecture of Hausel-Thaddeus [19, 20] for the moduli space of strongly parabolic Higgs bundles of rank two and three, with full flags, for any generic weights. Although the main theorem is proved only for rank at most three, most of the results are proved for any prime rank.


## 1. Introduction

The Hitchin system is an algebraic completely integrable system. Since it was introduced by Hitchin [21, 22] thirty years ago, it has been the subject of much interest, and it has turned out to have profound connections with several other areas of mathematics. Its basic ingredient is the moduli space $\mathcal{M}_{d}(G)$ of $G$-Higgs bundles $(V, \varphi)$ of fixed topological type $d$ on a closed Riemann surface $X$ for a connected complex reductive group $G$. Here $V$ is a holomorphic principal $G$-bundle on $X$ and $\varphi$ is a holomorphic 1-form with values in $\operatorname{Ad}(V)$. This moduli space is a holomorphic symplectic manifold carrying a hyperKähler metric. The integrable system is given by the Hitchin map $h: \mathcal{M}_{d}(G) \rightarrow \mathcal{A}$, where the Hitchin base $\mathcal{A}$ is an affine space whose dimension is half that of $\mathcal{M}_{d}(G)$ and the components of $h$ are the coefficients of the characteristic polynomial of $\varphi$.
Mirror symmetry for the Hitchin system was introduced in the work of Hausel and Thaddeus [20] (announced in [19]). It involves the Hitchin systems for the pair of Langlands dual groups $G=\operatorname{SL}(n, \mathbb{C})$ and ${ }^{L} G=\operatorname{PGL}(n, \mathbb{C})$. Hausel and Thaddeus proved that the moduli spaces are mirror partners in the sense of Strominger-Yau-Zaslow (SYZ) [38]; since they considered the case when $n$ and the degree $d=\operatorname{deg}(V)$ are coprime, this requires equipping the moduli spaces with natural $B$-fields, or gerbes. Hausel and Thaddeus have also shown that, in the cases $n=2,3$, the moduli spaces satisfy topological mirror symmetry, which is an identity of suitably defined stringy $E$-polynomials (these encode stringy Hodge numbers and again involve the $B$-field). Moreover, they conjectured that this holds for any $n$ and $d$ with $(n, d)=1$.

It was later proved by Donagi-Pantev [12] that, more generally, SYZ mirror symmetry is satisfied by the moduli spaces (or stacks) of $G$-Higgs bundles for any complex reductive group $G$. On the other hand, a very recent preprint by Groechenig-Wyss-Ziegler [16] uses $p$-adic integration to prove topological mirror symmetry in the case $G=\operatorname{SL}(n, \mathbb{C})$ for any $n$ and $d$ with $(n, d)=1$.

Parabolic Higgs bundles were introduced by Simpson [36] as the natural objects to consider for extending non-abelian Hodge theory to punctured Riemann surfaces. They

[^0]are pairs consisting of a parabolic bundle $V$, i.e., a vector bundle with weighted flags in the fibers over fixed marked points in $X$, and a Higgs field $\varphi$ with values in the parabolic endomorphisms of $V$.

The theory of parabolic Higgs bundles is in many ways analogous to that of usual Higgs bundles. In particular, there is a parabolic version of the Hitchin system, which goes back to the work of Bottacin [10], Markman [29] and Nasatyr-Steer [32]. Parabolic Higgs bundles have subsequently been studied by many authors; we merely point to Boden-Yokogawa [9] and Logares-Martens [28] as convenient references for parabolic Higgs bundles and the parabolic Hitchin system. We emphasize that in this paper we consider exclusively strongly parabolic Higgs bundles, meaning that the residue of the Higgs field at the marked points is nilpotent. These provide the most immediate generalization of the Hitchin system in that their moduli spaces are symplectic leaves of more general (Poisson) moduli spaces of (non-strongly) parabolic Higgs bundles.

In the announcement [19] Hausel and Thaddeus also consider the parabolic case, and outline a proof that SYZ mirror symmetry holds for any $n$. Moreover, they state that topological mirror symmetry holds for parabolic Higgs bundles in the case $G=\mathrm{SL}(n, \mathbb{C})$ with $n=2,3$, and conjecture that it should be true for any $n$. The main result of the present paper is a proof of this conjecture for $n=2,3$ (Theorem 3.13 below). For simplicity we restrict ourselves to the case of full flags, though our calculations of $E$ polynomials can in fact be carried through in the general case.

Our proof follows the basic strategy of Hausel and Thaddeus. It rests on the observation that it suffices to prove that certain contributions on each side are identical in order to conclude that the full stringy $E$-polynomials coincide for the $\operatorname{SL}(n, \mathbb{C})$ and $\operatorname{PGL}(n, \mathbb{C})$ moduli spaces. On the $\operatorname{PGL}(n, \mathbb{C})$-moduli space, the relevant contribution to the stringy $E$-polynomial comes from the fixed loci in the moduli space under the natural action of non-trivial elements of the group $\Gamma_{n}$ of $n$-torsion points of $\operatorname{Pic}^{0}(X)$. On the $\operatorname{SL}(n, \mathbb{C})$ moduli space, the relevant contribution is the part of the $E$-polynomial which is not invariant under the action of $\Gamma_{n}$, also known as the variant part, and which is determined by the variant part of the $E$-polynomial of certain fixed point subvarieties under the natural $\mathbb{C}^{*}$-action.

The description of the fixed loci of elements of $\Gamma_{n}$ is broadly parallel to that of [20] and essentially rests on the work of Narasimhan-Ramanan [31]. The result is that the fixed point loci are described in terms of Prym varieties of unramified covers of $X$ modulo the action of the Galois group. However, in the parabolic situation, it turns out that this action can be absorbed in the parabolic data, and this simplifies the arguments somewhat compared to the non-parabolic situation.

On the other hand, the fixed points of the $\mathbb{C}^{*}$-action are so-called Hodge bundles. These are Higgs bundles whose underlying vector bundle has a direct sum decomposition $V=V_{1} \oplus \cdots \oplus V_{l}$ with respect to which the Higgs field $\varphi$ has weight one. For rank $n=2,3$, it is known that only fixed loci consisting of Hodge bundles whose summands are all line bundles contribute to the variant $E$-polynomial, but the corresponding result for higher prime rank - that only $\mathbb{C}^{*}$-fixed loci of type $(1,1, \ldots, 1)$ contribute to the variant $E$ polynomial - is not known to be true. This is the only missing step for generalising our proof to any prime rank $n$, since our calculations are done for every such $n$ (this is completely analogous to the non-parabolic case as treated in [20]).

It turns out that the $B$-field does not play a very prominent role in the parabolic situation. Indeed, for SYZ mirror symmetry to hold in the strict sense, i.e., without a $B$-field, it is required that there be a Lagrangian section of the fibration, providing the natural base points of the abelian varieties which are the fibers of the integrable system. This is provided by the parabolic version of the Hitchin section. There is an isomorphism
between moduli spaces of parabolic Higgs bundles for any two different degrees (requiring an adjustment of the weights), as long as at least one of the flags are full. Using this isomorphism, one proves that the Hitchin section - thus the required Lagrangian section - exists for any degree; cf. Theorem 3.5. One thus also expects the usual (i.e. without the $B$-field correction) stringy $E$-polynomials to agree, and this is indeed what we show to be the case. On the other hand Biswas-Dey [7] have proved that the moduli spaces of parabolic Higgs bundles satisfy SYZ mirror symmetry also when equipped with the natural $B$-fields (analogously to the non-parabolic case considered in [20]). Thus one would expect topological mirror symmetry to hold also for the $B$-field twisted stringy $E$-polynomials and, indeed, calculations analogous to those carried out here and in [20] indicate that this is the case.

We assume throughout that $g \geqslant 2$. However, we note that for $g=0$ the $E$-polynomials trivially agree since the moduli spaces are identical. For $g=1$ we have chosen not to include the calculation of $E$-polynomials, both in order to avoid considering special cases and also because on the SYZ side mirror symmetry is currently known only for $g \geqslant 2$ (since otherwise the generic fibre of the Hitchin map may be singular, see Section 3.1).

Here is an outline of the contents of the paper. Section 2 reviews basic facts about parabolic Higgs bundles and their moduli. We also recall how in the parabolic setting moduli spaces for different degrees $d$ are isomorphic (with a change in parabolic weights). In Section 3 we recall the SYZ mirror symmetry result for the parabolic Hitchin system, review the stringy $E$-polynomials, describe the topological mirror symmetry conjecture of Hausel-Thaddeus, and state and prove our result. Section 4 is devoted to the calculation of the contribution to the variant part of the $E$-polynomial of the $\mathrm{SL}(n, \mathbb{C})$-moduli space arising only from the $\mathbb{C}^{*}$-fixed point loci of type $(1,1, \ldots, 1)$. In Section 5 we recall some classical results on Prym varieties of unramified covers. These are used in Section 6, where the contribution from the fixed point loci of non-trivial elements of $\Gamma_{n}$ to the stringy $E$-polynomial of $\operatorname{PGL}(n, \mathbb{C})$-moduli space is calculated.

Acknowledgments. We thank David Alfaya, Emilio Franco, Laura Fredrickson, Oscar García-Prada, Tomas Gómez, Tamás Hausel, Jochen Heinloth and Ana Peón-Nieto for useful discussions.

## 2. Parabolic Higgs bundles and their moduli

In this section we recall basic facts about parabolic Higgs bundles and their moduli spaces.
2.1. Parabolic vector bundles. Denote by $X$ a smooth projective curve over $\mathbb{C}$, and mark it with distinct points labeled by the divisor

$$
D=p_{1}+\cdots+p_{|D|},
$$

with $p_{i} \neq p_{j}$ for $i \neq j$ and where $|D|=\operatorname{deg} D$. Let $g$ be the genus of $X$ and assume $g \geqslant 2$. This data will be fixed throughout.

Parabolic vector bundles on $X$ associated to $D$, are vector bundles together with extra structure over each point of $D$.

Definition 2.1. A holomorphic parabolic vector bundle of rank $n$ on $X$, associated to the divisor $D$, is a holomorphic vector bundle $V$ of rank $n$ over $X$, endowed with a parabolic structure along $D$. By this is meant a collection of weighted flags of the fibers of $V$ over each point $p \in D$ :

$$
\begin{align*}
& V_{p}=V_{p, 1} \supsetneq V_{p, 2} \supsetneq \cdots \supsetneq V_{p, s_{p}} \supsetneq V_{p, s_{p}+1}=\{0\},  \tag{2.1}\\
& 0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{s_{p}}(p)<1,
\end{align*}
$$

where $s_{p}$ is an integer between 1 and $n$. The real number $\alpha_{i}(p) \in[0,1)$ is the weight of the subspace $V_{p, i}$. The multiplicity of the weight $\alpha_{i}(p)$ is the number $m_{i}(p)=\operatorname{dim}\left(V_{p, i} / V_{p, i+1}\right)$, thus $\sum_{i} m_{i}(p)=n$. The data given only by the flags over $D$ (i.e., without the weights) is called the quasi-parabolic structure of $V$. The parabolic structure is obtained from a quasiparabolic structure by specifying the weights. The type of the quasi-parabolic structure is $\boldsymbol{m}$, where $\boldsymbol{m}=\left(m_{1}(p), \ldots, m_{s_{p}}(p)\right)_{p \in D}$ is the collection of all multiplicities over all points of $D$. The type of the parabolic structure is $(\boldsymbol{m}, \boldsymbol{\alpha})$, with $\boldsymbol{\alpha}=\left(\alpha_{1}(p), \ldots, a_{s_{p}}(p)\right)_{p \in D}$ being the collection of all weights. The type of the parabolic vector bundle is ( $n, d, \boldsymbol{m}, \boldsymbol{\alpha}$ ), where $d=\operatorname{deg}(V)$ is its degree. Finally, a flag over a point $p \in D$ is full if $s_{p}=n$ or, equivalently, $m_{i}(p)=1$ for all $i$.

We shall denote a parabolic vector bundle by just $V$ whenever the parabolic structure is clear from the context.

Remark 2.2. Given a parabolic vector bundle $V$, with parabolic structure of type ( $\boldsymbol{m}, \boldsymbol{\alpha}$ ), and a line bundle $L$ the tensor product $V \otimes L$ acquires a parabolic structure, of the same type ( $\boldsymbol{m}, \boldsymbol{\alpha}$ ), in the obvious way, i.e., by taking the flags on $V \otimes L$ along $D$ induced by the ones of $V$, with the same weights. Except when explicitly mentioned to the contrary, this will be the parabolic structure we shall consider on $V \otimes L$. In fact, it corresponds to the general tensor product of parabolic bundles (see Yokogawa [41) in the particular case where $L$ has trivial parabolic structure.

Next we come to morphisms of parabolic bundles. These will be the vector bundle homomorphisms which preserve the parabolic structures; however these can be preserved in a week or a strong sense.

Definition 2.3. Let $V$ and $W$ be parabolic vector bundles whose parabolic structures are of type $(\boldsymbol{m}, \boldsymbol{\alpha})$ and $(\boldsymbol{l}, \boldsymbol{\beta})$ respectively, and let $\phi: V \rightarrow W$ be a holomorphic map. The map $\phi$ is called parabolic if we have, for all $p \in D$,

$$
\alpha_{i}(p)>\beta_{j}(p) \Longrightarrow \phi\left(V_{p, i}\right) \subseteq W_{p, j+1} .
$$

Denote by $\operatorname{ParHom}(V, W)$ the bundle of parabolic homomorphisms from $V$ to $W$ and, if $W=V$, write $\operatorname{ParEnd}(V)$ instead. The map $\phi$ is said strongly parabolic if

$$
\alpha_{i}(p) \geqslant \beta_{j}(p) \Longrightarrow \phi\left(V_{p, i}\right) \subseteq W_{p, j+1},
$$

for all $p \in D$. Denote by $\operatorname{SParHom}(V, W)$ the bundle of strongly parabolic homomorphisms from $V$ to $W$ and, if $W=V$, write $\operatorname{SParEnd}(V)$ instead.
2.2. Parabolic Higgs bundles. We shall need to consider parabolic Higgs bundles with various structure groups $G$. Indeed, there is a theory of parabolic $G$-Higgs bundles (see, for example, Biquard-García-Prada-Mundet [6] for a general notion for real reductive $G$ ) but, since we shall only need the groups $\operatorname{GL}(n, \mathbb{C}), \operatorname{SL}(n, \mathbb{C})$ and $\operatorname{PGL}(n, \mathbb{C})$, we can make the following ad hoc definitions.
Let $K=\Omega_{X}^{1}$ be the canonical bundle on $X$ and write $K(D)=K \otimes \mathcal{O}_{X}(D)$.
Definition 2.4. A strongly parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle is a pair $(V, \varphi)$, where $V$ is a parabolic bundle of rank $n$ and the Higgs field $\varphi: V \rightarrow V \otimes K(D)$ is a strongly parabolic homomorphism, i.e., $\varphi$ is a holomorphic section of $\operatorname{SParEnd}(V) \otimes K(D)$, where $V \otimes K(D)$ has the parabolic structure defined by $V$ (cf. Remark (2.2). The type of a parabolic GL $(n, \mathbb{C})$-Higgs bundle $(V, \varphi)$ is the type of the parabolic vector bundle $V$; cf. Definition 2.1.

Thus, in a strongly parabolic Higgs bundle $(V, \varphi)$, the Higgs field $\varphi$ is a meromorphic endomorphism valued one-form with at most simple poles along $p \in D$ and whose residue
at $p$ is nilpotent with respect to the flag. In other words, if the parabolic structure on $V$ is given by (2.1) then,

$$
\varphi\left(V_{p, i}\right) \subseteq V_{p, i+1} \otimes K(D)_{p}
$$

Remark 2.5. If we require $\varphi$ to be just parabolic, rather than strongly parabolic, we get the notion of parabolic Higgs bundle (for the structure groups considered). We shall, however, never use this notion in the present paper. Thus we shall frequently omit the adverb "strongly", but the reader should keep in mind that the Higgs field $\varphi$ is always required to be strongly parabolic.

If $V$ is a parabolic bundle of rank $n$, consider the determinant line bundle $\Lambda^{n} V$. Though this has a natural parabolic structure, in the following definition we ignore it and consider just the underlying line bundle.

Definition 2.6. Fix a holomorphic line bundle $\Lambda$ on $X$ of degree $d \in \mathbb{Z}$. A strongly parabolic $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle with fixed determinant $\Lambda$ is a pair $(V, \varphi)$, where $V$ is a parabolic bundle of rank $n$ such that $\Lambda^{n} V \cong \Lambda$, and where $\varphi \in H^{0}(X, \operatorname{SParEnd}(V) \otimes$ $K(D))$ is such that $\operatorname{tr}(\varphi) \equiv 0$.

Note that, strictly speaking, " $\operatorname{SL}(n, \mathbb{C})$-bundle" should only refer to the case where the line bundle $\Lambda$ is trivial, so we are committing a slight abuse of language here.

When there is no need to specify the structure group, or when it is clear from the context, we shall often make a further innocuous abuse of language and say simply (strongly) parabolic Higgs bundle.

In order to introduce strongly parabolic $\operatorname{PGL}(n, \mathbb{C})$-Higgs bundles, recall that any holomorphic $\operatorname{PGL}(n, \mathbb{C})$-bundle over the curve $X$ lifts to a holomorphic vector bundle $V \rightarrow X$, and that two such lifts $V$ and $V^{\prime}$ differ by tensoring by a line bundle.

Definition 2.7. A strongly parabolic $\mathrm{PGL}(n, \mathbb{C})$-Higgs bundle is an equivalence class $[(V, \varphi)]$ of strongly parabolic GL $(n, \mathbb{C})$-Higgs bundles, where $(V, \varphi)$ and $\left(V^{\prime}, \varphi^{\prime}\right)$ are considered equivalent if there is a line bundle $L$ such that $V^{\prime} \cong V \otimes L$, the parabolic structure of $V^{\prime}$ is the one obtained from $V$, and $\varphi^{\prime}=\varphi \otimes \mathrm{Id}_{L}$.

Remark 2.8. Recall that $\operatorname{PGL}(n, \mathbb{C})$-bundles over the curve $X$ are topologically classified by $\pi_{1}(\operatorname{PGL}(n, \mathbb{C})) \cong \mathbb{Z}_{n}$. Fixing a topological type $c \in \mathbb{Z}_{n}$ and a holomorphic line bundle $\Lambda$ whose degree modulo $n$ equals $c$, any holomorphic $\operatorname{PGL}(n, \mathbb{C})$-bundle of topological type $c$ may be lifted to a holomorphic vector bundle whose determinant bundle is isomorphic to $\Lambda$. Moreover, two lifts with the same determinant bundle differ by tensoring by a line bundle which is a $n$-torsion point of the $\operatorname{Jacobian} \operatorname{Jac}(X)$. These facts reflect the identifications $\operatorname{PGL}(n, \mathbb{C})=\operatorname{GL}(n, \mathbb{C}) / \mathbb{C}^{*}=\operatorname{SL}(n, \mathbb{C}) / \mathbb{Z}_{n}=\operatorname{PSL}(n, \mathbb{C})$.
2.3. Stability and moduli spaces. In the following we recall the stability condition for parabolic Higgs bundles and introduce their moduli spaces.

Definition 2.9. Given a parabolic vector bundle $V$, a parabolic subbundle is a vector subbundle $V^{\prime} \subseteq V$, with the parabolic structure defined as follows. For each $p \in D$, the quasi-parabolic structure is given by the flag

$$
V_{p}^{\prime}=V_{p, 1}^{\prime} \supsetneq V_{p, 2}^{\prime} \supsetneq \cdots \supsetneq V_{p, s_{p}^{\prime}}^{\prime} \supsetneq\{0\},
$$

where $V_{p, i}^{\prime}=V_{p}^{\prime} \cap V_{p, i}$, discarding all the repetitions of subspaces in the filtration. Moreover, the weights $0 \leqslant \alpha_{1}^{\prime}(p)<\cdots<\alpha_{s_{p}^{\prime}}^{\prime}(p)<1$ are taken to be the greatest possible among the corresponding original weights, meaning that

$$
\begin{equation*}
\alpha_{i}^{\prime}(p)=\max _{j}\left\{\alpha_{j}(p) \mid V_{p, j} \cap V_{p}^{\prime}=V_{p, i}^{\prime}\right\}=\max _{j}\left\{\alpha_{j}(p) \mid V_{p, i}^{\prime} \subseteq V_{p, j}\right\} . \tag{2.2}
\end{equation*}
$$

In other words, the weight attached to $V_{p, i}^{\prime}$ is the weight $\alpha_{j}(p)$ whose index $j$ is such that $V_{p, i}^{\prime} \subseteq V_{p, j}$ but $V_{p, i}^{\prime} \nsubseteq V_{p, j+1}$.
Definition 2.10. The degree of a parabolic Higgs bundle $(V, \varphi)$ is the degree of the underlying bundle, $\operatorname{deg}(V) \in \mathbb{Z}$. The parabolic degree $\operatorname{pardeg}(V)$ and parabolic slope $\operatorname{par} \mu(V)$ of $(V, \varphi)$ are the parabolic degree and slope, respectively, of the underlying parabolic vector bundle, defined by

$$
\operatorname{pardeg}(V)=\operatorname{deg}(V)+\sum_{p \in D} \sum_{i=1}^{s_{p}} m_{i}(p) \alpha_{i}(p) \quad \text { and } \quad \operatorname{par} \mu(V)=\frac{\operatorname{pardeg}(V)}{n} .
$$

Definition 2.11. A strongly parabolic Higgs bundle $(V, \varphi)$ is semistable if

$$
\operatorname{par} \mu\left(V^{\prime}\right) \leqslant \operatorname{par} \mu(V)
$$

for every non-zero parabolic subbundle $V^{\prime} \subseteq V$ which is $\varphi$-invariant, that is, $\varphi\left(V^{\prime}\right) \subseteq$ $V^{\prime} \otimes K(D)$. It is stable if it is semistable and strict inequality holds above for all proper non-zero $\varphi$-invariant parabolic subbundles $V^{\prime} \subseteq V$.

Consider now quasi parabolic Higgs bundles of rank $n$, degree $d$, and quasi-parabolic type $\boldsymbol{m}$. The space of compatible parabolic weights $\boldsymbol{\alpha}=\left(\alpha_{1}(p), \ldots, \alpha_{s_{p}}(p)\right)_{p \in D}$ is a product $\mathcal{S}$ of simplices (excluding some boundaries) determined by the inequalities in (2.1), one simplex for each point of $D$. Let $(V, \varphi)$ be a parabolic Higgs bundle of type $(\boldsymbol{m}, \boldsymbol{\alpha})$. If $(V, \varphi)$ is semistable but not stable, then

$$
\begin{equation*}
n\left(d^{\prime}+\sum_{p \in D} \sum_{i=1}^{s_{p}^{\prime}} m_{i}^{\prime}(p) \alpha_{i}^{\prime}(p)\right)=n^{\prime}\left(d+\sum_{p \in D} \sum_{i=1}^{s_{p}} m_{i}(p) \alpha_{i}(p)\right), \tag{2.3}
\end{equation*}
$$

where $d^{\prime}$ and $n^{\prime}$ are the degree and rank of a destabilizing parabolic Higgs subbundle $\left(V^{\prime},\left.\varphi\right|_{V^{\prime}}\right)$ with multiplicities $m_{i}^{\prime}(p)$ and weights $\alpha_{i}^{\prime}(p)$. For given $d^{\prime}, n^{\prime}$ and $m_{i}^{\prime}(p)$, equation (2.3) determines an intersection of a hyperplane with $\mathcal{S}$ which is called a wall. There are only finitely many possible values for $n^{\prime}$ and $m_{i}^{\prime}(p)$ and, for each of these, there are also finitely many values of $d^{\prime}$ for which these hyperplanes intersect $\mathcal{S}$. Hence there are finitely many walls.

Definition 2.12. Fix $n, d$ and a quasi-parabolic type $\boldsymbol{m}$. A weight vector $\boldsymbol{\alpha} \in \mathcal{S}$ is called generic if it does not belong to a wall. A connected component of the complement of the set of walls is called a chamber.

Remark 2.13. It is immediate from this definition that for generic weights, a semistable parabolic Higgs bundle is in fact stable. Moreover, for generic weights in the same chamber, the stability condition is unchanged, so the corresponding moduli spaces (to be introduced presently) will be isomorphic.

A GIT construction of the moduli space $\mathcal{M}_{d}^{m, \alpha}(\mathrm{GL}(n, \mathbb{C}))$ of semistable parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles over $X$, of rank $n$, degree $d$ and parabolic type ( $\boldsymbol{m}, \boldsymbol{\alpha}$ ), was carried out by Yokogawa [40], and the deformation theory of parabolic Higgs bundles was also studied by Yokogawa [41] (cf. Boden-Yokogawa [9]). A gauge theoretic construction of the moduli space of (non-strongly) parabolic Higgs bundles was done by Konno [26]. It was proved by Yokogawa that the stable locus of the moduli space is smooth and quasi-projective. Thus we have the following result.

Proposition 2.14. Assume that the weights $\boldsymbol{\alpha}$ are generic. Then the moduli space $\mathcal{M}_{d}^{m, \alpha}(\mathrm{GL}(n, \mathbb{C}))$ is a smooth quasi-projective variety. Moreover, for generic weights in the same chamber of $\mathcal{S}$, the corresponding moduli spaces are isomorphic.

In order to obtain the moduli space of parabolic $\operatorname{SL}(n, \mathbb{C})$-Higgs bundles over $X$, consider the determinant map

$$
\begin{aligned}
p: \mathcal{M}_{d}^{m, \alpha}(\mathrm{GL}(n, \mathbb{C})) & \rightarrow T^{*} \operatorname{Pic}^{d}(X) \cong \operatorname{Pic}^{d}(X) \times H^{0}(X, K), \\
(V, \varphi) & \mapsto\left(\Lambda^{n} V, \operatorname{tr}(\varphi)\right),
\end{aligned}
$$

with $\operatorname{Pic}^{d}(X)$ the component of the Picard variety of $X$ of degree $d$ line bundles. Notice that, since $(V, \varphi)$ is strongly parabolic, the residue of the $\operatorname{trace} \operatorname{tr}(\varphi)$ vanishes along $D$, and so it is in fact a section of $K$. Let

$$
\mathcal{M}_{\Lambda}^{m, \alpha}(\operatorname{SL}(n, \mathbb{C}))=p^{-1}(\Lambda, 0)
$$

For generic weights, this is again a smooth quasi-projective variety. Since this is the moduli space we shall mostly be working with, whenever there is no risk of confusion, we shall denote it simply by $\mathcal{M}$.

Next we want to introduce the moduli space of parabolic $\operatorname{PGL}(n, \mathbb{C})$-Higgs bundles. In view of Definition 2.7 and Remark 2.8 we consider the group

$$
\Gamma_{n}=\operatorname{Jac}_{[n]}(X)=\left\{L \in \operatorname{Jac}(X) \mid L^{n} \cong \mathcal{O}_{X}\right\} \subset \operatorname{Jac}(X)
$$

of $n$-torsion points of the Jacobian of $X$. Recall that $\Gamma_{n} \cong H^{1}\left(X, \mathbb{Z}_{n}\right) \cong \mathbb{Z}_{n}^{2 g}$. It will be convenient to distinguish the elements of $\Gamma_{n}$ as an abstract group and as line bundles; thus, if $\gamma$ denotes an element of $\Gamma_{n}$, the corresponding line bundle will be denoted by $L_{\gamma}$. Fix a line bundle $\Lambda$ and let $[d] \in \mathbb{Z}_{n}$ denote the reduction of $d=\operatorname{deg}(\Lambda)$ modulo $n$. The group $\Gamma_{n}$ acts on $\mathcal{M}$ by

$$
\begin{equation*}
\gamma \cdot(V, \varphi)=\left(V \otimes L_{\gamma}, \varphi \otimes \operatorname{Id}_{L_{\gamma}}\right) \tag{2.4}
\end{equation*}
$$

(note that $\Gamma_{n}$ acts trivially on the parabolic structure). We take the moduli space of parabolic $\operatorname{PGL}(n, \mathbb{C})$-Higgs bundles of topological type $[d]$ to be

$$
\mathcal{M}_{[d]}^{m, \alpha}(\operatorname{PGL}(n, \mathbb{C}))=\mathcal{M} / \Gamma_{n}
$$

We remark that this is consistent with the abstract definition of stability of parabolic $G$-Higgs bundles coming from [6]. As opposed to $\mathcal{M}$, the moduli space $\mathcal{M} / \Gamma_{n}$ is not smooth, but rather an orbifold, with singularities arising from the fixed points of the action of $\Gamma_{n}$.

Serre duality for a parabolic vector bundles (see [41, 9]) says that for a parabolic vector bundle $V$

$$
H^{1}(\operatorname{ParEnd}(V)) \simeq H^{0}(\operatorname{SParEnd}(V) \otimes K(D))^{*}
$$

(and analogously in the traceless case), in other words, the infinitesimal deformation space of $V$ is dual to the space of Higgs fields on $V$. Thus, letting $\mathcal{N}$ denote the moduli space of parabolic vector bundles (with fixed determinant $\Lambda$ ), there is an embedding of the cotangent bundle $T^{*} \mathcal{N} \hookrightarrow \mathcal{M}$ as an open subset. The natural symplectic structure on the cotangent bundle extends to $\mathcal{M}$, which is thus a holomorphic symplectic manifold (see Bottacin [10, Sec. 5], Biswas-Ramanan [8, Sec. 6], Konno [26], Logares-Martens [28], and cf. Yokogawa [41] and Boden-Yokogawa [9]). Moreover, Konno's gauge theoretic construction (loc. cit.) shows that $\mathcal{M}$ has a compatible hyper-Kähler metric.
2.4. Isomorphism between moduli spaces for different degrees and weights. Let $\Lambda$ and $\Lambda^{\prime}$ be line bundles on $X$, not necessarily of the same degree. In this section we prove that, under mild conditions on the parabolic structure $\boldsymbol{\alpha}$, one can find a parabolic structure $\boldsymbol{\alpha}^{\prime}$ so that the moduli spaces $\mathcal{M}_{\Lambda}^{\alpha}(\operatorname{SL}(n, \mathbb{C}))$ and $\mathcal{M}_{\Lambda^{\prime}}^{\alpha^{\prime}}(\mathrm{SL}(n, \mathbb{C}))$ are isomorphic. This expands on [13, Proposition 2.1] and marks a substantial difference to the nonparabolic case, where such an isomorphism can only exist if $\operatorname{deg}(\Lambda)$ and $\operatorname{deg}\left(\Lambda^{\prime}\right)$ are equal modulo $n$.

We need the notion of tensor product of parabolic (Higgs) bundles. This is better viewed in the more general context of parabolic (or filtered) sheaves (see Boden-Yokogawa [9], Yokogawa [41], and Simpson [36]) but we shall only need a few simple facts which we now review. In fact it suffices for us to consider the case when one of the bundles is a parabolic line bundle (with trivial Higgs field), so let ( $V, \boldsymbol{\alpha}$ ) be a parabolic vector bundle bundle of rank $n$ and let $(L, \boldsymbol{\beta})$ be a parabolic line bundle on $X$. There is a parabolic tensor product of the parabolic bundles $V$ and $L$, denoted by $\left(V \otimes^{P} L, \boldsymbol{\alpha}^{\prime}\right)$. The parabolic weights $\boldsymbol{\alpha}^{\prime}$ are given by

$$
\alpha_{i}^{\prime}(p)= \begin{cases}\alpha_{i}(p)+\beta(p) & \text { if } \alpha_{i}(p)+\beta(p)<1  \tag{2.5}\\ \alpha_{i}(p)+\beta(p)-1 & \text { if } \alpha_{i}(p)+\beta(p) \geqslant 1\end{cases}
$$

where for each $p \in D$ the correct ordering of the $\alpha_{i}^{\prime}(p)$ by size corresponds to a cyclic permutation of the ordering of the indices $i=1, \ldots, s_{p}$. The multiplicity of the weight $\alpha_{i}^{\prime}(p)$ is $m_{i}(p)$. The parabolic degree of the parabolic tensor product is given by the usual formula:

$$
\begin{aligned}
\operatorname{pardeg}\left(V \otimes^{P} L\right) & =\operatorname{pardeg}(V)+n \operatorname{pardeg}(L) \\
& =\operatorname{deg}(V)+\sum_{p, i} m_{i}(p) \alpha_{i}(p)+n\left(\operatorname{deg}(L)+\sum_{p} \beta(p)\right) .
\end{aligned}
$$

In view of this we get the following formula for the (non-parabolic) degree of $V \otimes^{P} L$ :

$$
\begin{align*}
\operatorname{deg}\left(V \otimes^{P} L\right) & =\operatorname{pardeg}\left(V \otimes^{P} L\right)-\sum_{p, i} m_{i}(p) \alpha_{i}^{\prime}(p)  \tag{2.6}\\
& =\operatorname{deg}(V)+n \operatorname{deg}(L)+\sum_{p, i} m_{i}(p)\left(\alpha_{i}(p)+\beta(p)-\alpha_{i}^{\prime}(p)\right)
\end{align*}
$$

Finally we remark that if $V$ underlies a parabolic Higgs bundle $(V, \varphi)$, then $V \otimes^{P} L$ has a Higgs field induced by $\varphi \otimes \operatorname{Id}_{L}$ and that $(V, \varphi)$ is stable if and only if $\left(V \otimes^{P} L, \varphi \otimes \operatorname{Id}_{L}\right)$ is (see Simpson [36]).

Theorem 2.15. Let $\boldsymbol{\alpha}=\left(\alpha_{i}(p)\right)_{p \in D}$ be a system of parabolic weights and let $\Lambda$ and $\Lambda^{\prime}$ be line bundles of degrees $d$ and $d^{\prime}$, respectively. Suppose there are $\boldsymbol{\beta}=(\beta(p))_{p \in D}$ in $[0,1)$ such that

$$
\begin{equation*}
d^{\prime}-d-\sum_{p, i} m_{i}(p)\left(\alpha_{i}(p)+\beta(p)-\alpha_{i}^{\prime}(p)\right) \equiv 0 \quad(\bmod n) \tag{2.7}
\end{equation*}
$$

Then there is a parabolic line bundle $(L, \boldsymbol{\beta})$ inducing an isomorphism

$$
\begin{aligned}
\mathcal{M}_{\Lambda}^{\alpha}(\mathrm{SL}(n, \mathbb{C})) & \cong \mathcal{M}_{\Lambda^{\prime}}^{\alpha^{\prime}}(\mathrm{SL}(n, \mathbb{C})), \\
(V, \varphi) & \mapsto\left(V \otimes^{P} L, \varphi \otimes \mathrm{Id}_{L}\right),
\end{aligned}
$$

where the weights $\boldsymbol{\alpha}^{\prime}$ are given by (2.5).
Proof. In view of (2.6) and the remarks preceding the statement of the theorem, we can find a line bundle $L$ such that tensoring by $L$ gives an isomorphism $\mathcal{M}_{d}^{m, \alpha}(\mathrm{GL}(n, \mathbb{C})) \stackrel{\cong}{\leftrightarrows}$ $\mathcal{M}_{d^{\prime}}^{m, \alpha^{\prime}}(\mathrm{GL}(n, \mathbb{C}))$. In order to get the isomorphism between the fixed determinant moduli spaces it suffices to adjust $L$ by tensoring by a suitable (non-parabolic) degree zero line bundle.

The following corollary describes two situations where it is possible to find $\boldsymbol{\beta}$ satisfying (2.7). The conditions imposed are mild, and since we will at some point impose full flags at all points of $D$, we will be under the assumptions of this corollary.

Corollary 2.16. Consider the moduli space $\mathcal{M}_{\Lambda}^{m, \alpha}(\operatorname{SL}(n, \mathbb{C}))$.
(1) If $\Lambda^{\prime}$ is any line bundle of degree $d^{\prime}$ such that $d^{\prime} \equiv d(\bmod n)$, then $\mathcal{M}_{\Lambda}^{m, \alpha}(\operatorname{SL}(n, \mathbb{C})) \cong$ $\mathcal{M}_{\Lambda^{\prime}}^{m, \alpha}(\operatorname{SL}(n, \mathbb{C}))$.
(2) Suppose the parabolic structure $\boldsymbol{\alpha}$ is such that for some point $p_{0} \in D$ the flag is full (i.e., $s_{p_{0}}=n$ ). Then, given any line bundle $\Lambda^{\prime}$ of any degree $d^{\prime}$, there exists a parabolic structure $\boldsymbol{\alpha}^{\prime}$ such that $\mathcal{M}_{\Lambda^{\prime}}^{m, \boldsymbol{\alpha}^{\prime}}(\mathrm{SL}(n, \mathbb{C})) \cong \mathcal{M}_{\Lambda}^{m, \boldsymbol{\alpha}}(\mathrm{SL}(n, \mathbb{C}))$.
Proof. For the first item, we just have to take $L$ a $n$th root of $\Lambda^{\prime} \Lambda^{-1}$ and use the usual tensor product $V \mapsto V \otimes L$. This is of course the generalization to the parabolic case of the classical isomorphism in the non-parabolic case.

For the second item, suppose that $d^{\prime}-d \equiv k(\bmod n)$. Since the flag over $p_{0}$ is full, we can choose $\beta\left(p_{0}\right) \in[0,1)$ such that $\#\left\{i \mid \alpha_{i}\left(p_{0}\right)+\beta\left(p_{0}\right) \geqslant 1\right\}=k$. If $p \in D \backslash\left\{p_{0}\right\}$, take $\beta(p)=0$, so $\#\left\{i \mid \alpha_{i}(p)+\beta(p) \geqslant 1\right\}=0$. With these choices, (2.7) holds in view of (2.5), and the conclusion follows by the theorem.
2.5. Basic assumptions. We now make two assumptions.

Assumption 2.17. We assume from now on that:
(1) the weights $\boldsymbol{\alpha}$ are generic;
(2) the flags over all points of $D$ are full (i.e., $m_{i}(p)=1$ for all $i$ and all $p \in D$, thus $s_{p}=n$ for all $p$ ).

Since from now on $m_{i}(p)=1$ for all $i, p$, we shall remove the $\boldsymbol{m}$ from the notation.
The first assumption is essential for us. It implies that any semistable parabolic Higgs bundle is stable and hence, as shown by Yokogawa [40], the moduli space is smooth.

The second assumption serves two purposes. Firstly, the SYZ mirror symmetry picture (outlined in the next section) has currently only been shown under this assumption. Secondly, it simplifies the formulas in our calculations of Hodge polynomials. We point out, however, that these calculations generalize without too much trouble to the case of general flags.

Summarizing, under Assumption 2.17, the moduli space of parabolic Higgs bundles $\mathcal{M}$ is a smooth quasi-projective hyper-Kähler manifold. Its dimension can be calculated using deformation theory (see, for example, [13, Proposition 2.4]) and is given by

$$
\begin{equation*}
\operatorname{dim}(\mathcal{M})=2\left(n^{2}-1\right)(g-1)+|D| n(n-1), \tag{2.8}
\end{equation*}
$$

where we recall that $|D|=\operatorname{deg}(D)$ is the number of marked points on $X$.

## 3. Mirror symmetry

In this section we recall the Hausel-Thaddeus mirror symmetry proposal in the parabolic case. First, in Section 3.1 we treat the Hitchin system and mirror symmetry according to Strominger-Yau-Zaslow. Next, in Section 3.2, we recall the definition of the stringy $E$-polynomial and show, following Thaddeus, its independence of the parabolic weights. Finally, in Section 3.3 we state our main result and outline its proof.
3.1. The Hitchin map and SYZ mirror symmetry. In this section we briefly describe how $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$ are mirror partners in the sense of Strominger-Yau-Zaslow (SYZ) [38. This has been shown in the parabolic case by Biswas-Dey [7] (following Hausel-Thaddeus [20]). The general version of SYZ mirror symmetry proved by these authors requires considering a naturally defined gerbe (or $B$-field) on the moduli spaces (see also Donagi-Pantev [12]). As explained below, the statement of SYZ mirror symmetry involves identifying fibers of the Hitchin maps of the two moduli spaces as dual abelian varieties. The need for introducing the $B$-field comes from the lack of a natural
base point in these fibers. In the parabolic case there is a twist in the story: the moduli spaces $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$ are also mirror partners in the original sense of SYZ.

Remark 3.1. We recall that the true mirror partners are in fact the de Rham moduli spaces; these are moduli spaces of local systems on $X$ and are diffeomorphic to the Higgs bundle moduli spaces under the non-abelian Hodge correspondence. As explained in [20, Sec. 1] in the non-parabolic case, the statements on the de Rham side can be translated into statements on the Higgs bundle side through a hyper-Kähler rotation, and this works exactly the same way in the parabolic case. We refer the reader to Simpson [36] and Alfaya-Gómez [1] for details on the de Rham moduli spaces in the parabolic case.

We now introduce the Hitchin system in the parabolic setting. This goes back to Bottacin [10] and Nasatyr-Steer [32]. We start by defining the Hitchin map $h$ on the moduli spaces $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$ : it takes a parabolic Higgs bundle $(V, \varphi)$ to the coefficients of the characteristic polynomial of the twisted endomorphism $\varphi: V \rightarrow V \otimes K(D)$. Thus $h(V, \varphi)=\left(s_{2}, \ldots, s_{n}\right)$, with $s_{i}=\operatorname{tr}\left(\wedge^{i} \varphi\right)$. Since $\varphi$ is strongly parabolic, its restriction to every $p \in D$ is nilpotent, and so all the corresponding coefficients $s_{i}(p)$ of the characteristic polynomial vanish. We therefore have

$$
\begin{align*}
& h: \mathcal{M} \rightarrow \mathcal{A}=\bigoplus_{i=2}^{n} H^{0}\left(X, K^{i}((i-1) D)\right),  \tag{3.1}\\
& (V, \varphi) \mapsto\left(s_{2}, \ldots, s_{n}\right),
\end{align*}
$$

where $\mathcal{A}$ is the Hitchin base. It is clear that $h$ factors through the quotient $\mathcal{M} / \Gamma_{n}$, so we also have a Hitchin map $h^{\prime}$ on this moduli space:


Observe that

$$
\operatorname{dim}(\mathcal{A})=\left(n^{2}-1\right)(g-1)+\frac{n(n-1)|D|}{2}=\frac{\operatorname{dim}(\mathcal{M})}{2}
$$

By 40, the map $h$ is proper, hence so is $h^{\prime}$. The coordinate functions of $h$ and $h^{\prime}$ are independent and Poisson commute, and these maps form the Hitchin systems for $\operatorname{SL}(n, \mathbb{C})$ and $\operatorname{PGL}(n, \mathbb{C})$, respectively. In particular, for $s \in \mathcal{A}$, the fibers $h^{-1}(s)$ and $h^{\prime-1}(s)$ are complex Lagrangian subvarieties of $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$.

To describe the generic Hitchin fibers more precisely, consider the quasi-projective surface given by the total space $|K(D)|$ of $K(D)$ and the projection $\pi:|K(D)| \rightarrow X$. Given a point $s=\left(s_{2}, \ldots, s_{n}\right)$ in the Hitchin base $\mathcal{A}$, there is a projective curve $X_{s}$, lying in $|K(D)|$, defined by the zeros of the section

$$
\begin{equation*}
\lambda^{n}+\pi^{*} s_{2} \lambda^{n-2}+\cdots+\pi^{*} s_{n} \in H^{0}\left(|K(D)|, \pi^{*}\left(K^{n}(n D)\right)\right) \tag{3.2}
\end{equation*}
$$

where $\lambda$ is the tautological section of $\pi^{*}(K(D))$ and $\lambda^{n-i} \pi^{*} s_{i} \in H^{0}\left(|K(D)|, \pi^{*}\left(K^{n}((n-\right.\right.$ 1) $D))) \subseteq H^{0}\left(|K(D)|, \pi^{*}\left(K^{n}(n D)\right)\right)$. The curve $X_{s}$ is called the spectral curve associated to $s \in \mathcal{A}$. The restriction of $\pi$ to $X_{s}$ gives an $n$-cover $\pi: X_{s} \rightarrow X$ which is ramified over the locus where (3.2) has multiple roots. This locus is always non-empty.

By Lemma 3.1 of [14], there is an open and dense subspace $\mathcal{A}^{\prime} \subset \mathcal{A}$ such that $X_{s}$ is smooth whenever $s \in \mathcal{A}^{\prime}$ (this needs the assumption $g \geqslant 2$ on the genus of $X$ ). Moreover, for such generic $s$, Lemma 3.2 of [14] states that the fibre $h^{-1}(s)$ is naturally isomorphic to

$$
\begin{equation*}
P^{d^{\prime}}=\left\{L \in \operatorname{Pic}^{d^{\prime}}\left(X_{s}\right) \mid \operatorname{det}\left(\pi_{*} L\right) \cong \Lambda\right\}, \tag{3.3}
\end{equation*}
$$

where $d^{\prime}=d+n(n-1)(g-1+|D| / 2)$.
Remark 3.2. Lemma 3.2 of [14] needs the full flags assumption on every point of the divisor $D$. This is one reason why we confine ourselves to the full flag condition.

Definition 3.3. Consider a degree $n$ cover $\pi: Y \rightarrow X$. The norm map between the groups of divisors $\mathrm{Nm}_{\pi}: \operatorname{Div}(Y) \rightarrow \operatorname{Div}(X)$ is the homomorphism taking a divisor $E=$ $\sum E(p) p$ on $Y$ to the divisor $\operatorname{Nm}_{\pi}(E)=\sum E(p) \pi(p)$ on $X$.

The norm map just defined factors through the norm map between the Picard groups $\mathrm{Nm}_{\pi}: \operatorname{Pic}^{0}\left(X_{s}\right) \rightarrow \operatorname{Pic}^{0}(X)$, by $\operatorname{Nm}_{\pi}(L)=L^{\prime}$ where $L \cong \mathcal{O}_{Y}(E)$ and $L^{\prime} \cong \mathcal{O}_{X}\left(\operatorname{Nm}_{\pi}(E)\right)$.

Definition 3.4. The Prym variety of $Y$ associated to $\pi$, denoted by $\operatorname{Prym}_{\pi}(Y)$, is the abelian subvariety of $\operatorname{Pic}^{0}(Y)$ defined as the identity component of the kernel of $\mathrm{Nm}_{\pi}$.

The kernel of $\mathrm{Nm}_{\pi}$ is connected if and only if $\pi$ is ramified and sometimes the term Prym variety is used for the full kernel of $\mathrm{Nm}_{\pi}$. We have adopted Definition 3.4 in accordance with [20]. Since for $s \in \mathcal{A}^{\prime}$ the cover $\pi: X_{s} \rightarrow X$ is ramified], we have

$$
\begin{equation*}
\operatorname{Prym}_{\pi}\left(X_{s}\right)=\operatorname{ker}\left(\operatorname{Nm}_{\pi}\right)=\left\{L \in \operatorname{Pic}^{0}\left(X_{s}\right) \mid \operatorname{Nm}_{\pi}(L) \cong \mathcal{O}_{X}\right\} \tag{3.4}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\operatorname{det}\left(\pi_{*} L\right) \cong \operatorname{Nm}_{\pi}(L) \otimes \operatorname{det}\left(\pi_{*}\left(\mathcal{O}_{X_{s}}\right)\right) \cong \operatorname{Nm}_{\pi}(L) \otimes(K(D))^{-n(n-1) / 2} \tag{3.5}
\end{equation*}
$$

Thus we see that

$$
P^{d^{\prime}} \cong\left\{L \in \operatorname{Pic}^{d^{\prime}}\left(X_{s}\right) \mid \operatorname{Nm}_{\pi}(L) \cong \Lambda(K(D))^{n(n-1) / 2}\right\}
$$

is a torsor for $\operatorname{Prym}_{\pi}\left(X_{s}\right)$.
It is also easy to see that $h^{\prime-1}(s)$ is isomorphic $P^{d^{\prime}} / \Gamma_{n}$, hence it is a torsor for the quotient $\operatorname{Prym}_{\pi}\left(X_{s}\right) / \Gamma_{n}$, where $\Gamma_{n}$ acts by tensoring by the pull-back via $\pi$. The quotient $\operatorname{Prym}_{\pi}\left(X_{s}\right) / \Gamma_{n}$ is an abelian variety, isogenous to $\operatorname{Prym}_{\pi}\left(X_{s}\right)$.

By dualising the short exact sequence coming from the norm map, one easily checks [20, Lemma 2.3] that these two abelian varieties are dual to each other, in the sense that

$$
\operatorname{Pic}^{0}\left(\operatorname{Prym}_{\pi}\left(X_{s}\right)\right) \cong \operatorname{Prym}_{\pi}\left(X_{s}\right) / \Gamma_{n} \quad \text { and } \quad \operatorname{Pic}^{0}\left(\operatorname{Prym}_{\pi}\left(X_{s}\right) / \Gamma_{n}\right) \cong \operatorname{Prym}_{\pi}\left(X_{s}\right)
$$

Theorem 3.5 (Hausel-Thaddeus [19]). Assume that $s \in \mathcal{A}$ has simple zeros. Then the Hitchin fibers $h^{-1}(s)$ and $h^{\prime-1}(s)$ can be naturally identified with a pair of dual abelian varieties. Hence $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$ are SYZ mirror partners.
Proof. Assume first that $\Lambda \cong K(D)^{-n(n-1) / 2}$ (so that $d=-n(n-1)(g-1+|D| / 2)$ and $d^{\prime}=0$ ). In this case, (3.3), (3.4) and (3.5) show that the fibre of the Hitchin map $h: \mathcal{M} \rightarrow \mathcal{A}$ over a generic point $s \in \mathcal{A}^{\prime}$ is naturally identified with $\operatorname{Prym}_{\pi}\left(X_{s}\right)$ and not just a torsor over it, and analogously for the fibre of $h^{\prime}$. Hence, in view of the observations preceding the statement of the theorem, we have the desired conclusion if we show that the base points of these Pryms form a Lagrangian section $\sigma$ of $h$.

This is obtained, similarly to Hitchin's construction [23] in the non-parabolic case, as follow $\mathbb{2}^{2}$. Given $\left(s_{2}, \ldots, s_{n}\right) \in \mathcal{A}$, consider the parabolic Higgs bundle $(V, \varphi)$ where

$$
\begin{equation*}
V=K(D)^{1-n} \oplus \cdots \oplus K(D)^{-1} \oplus \mathcal{O}_{X} \tag{3.6}
\end{equation*}
$$

[^1]with the Higgs field defined, in this decomposition, by
\[

\varphi=\left($$
\begin{array}{ccccc}
0 & 1 & \cdots & 0 & 0  \tag{3.7}\\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 \\
s_{n} & s_{n-1} & \cdots & s_{2} & 0
\end{array}
$$\right)
\]

and such that the full flag of $V_{p}$ over each $p \in D$ is given by

$$
\begin{equation*}
V_{p} \supsetneq K(D)_{p}^{2-n} \oplus \cdots \oplus \mathcal{O}_{X, p} \supsetneq \cdots \supsetneq \mathcal{O}_{X, p} \supsetneq\{0\}, \tag{3.8}
\end{equation*}
$$

with the weights determined by $\mathcal{M}$. It is clear that $\varphi$ is strongly parabolic. Moreover, for any choice of weights $\boldsymbol{\alpha}$, the stability of $(V, \varphi)$ follows by a straightforward adaptation of the corresponding argument in [23]. Hence the map $\sigma: \mathcal{A} \rightarrow \mathcal{M},\left(s_{2}, \ldots, s_{n}\right) \mapsto(V, \varphi)$, where the parabolic $\operatorname{Higgs}$ bundle $(V, \varphi)$ is defined by (3.6) $-(3.8)$, provides a section of $h$. This section is Lagrangian for the same reason as in the non-parabolic case, namely that the underlying bundle is fixed as $\left(s_{2}, \ldots, s_{n}\right)$ moves in the Hitchin base. Finally, for each $s \in \mathcal{A}^{\prime}, V$ in (3.6) is such that $V \cong \pi_{*} \mathcal{O}_{X_{s}}$, so $\sigma$ is indeed a Lagrangian section through the base points $\mathcal{O}_{X_{s}} \in \operatorname{Prym}_{\pi}\left(X_{s}\right)$.
Now, for any other line bundle $\Lambda$ of any degree, use Corollary 2.16 to get an isomorphism $\mathcal{M}_{\Lambda}^{\alpha^{\prime}}(\mathrm{SL}(n, \mathbb{C})) \cong \mathcal{M}_{K(D)^{-n(n-1) / 2}}^{\alpha}(\mathrm{SL}(n, \mathbb{C}))$ (since we are assuming full flags, the hypotheses of Corollary 2.16 are satisfied). Clearly this map is in fact an isomorphism of the corresponding Hitchin systems, and descends to the PGL $(n, \mathbb{C})$-Hitchin systems, giving us in particular the desired identifications of the Hitchin fibers as dual abelian varieties.

Remark 3.6. This fits with a general phenomenon in SYZ-mirror symmetry, where if torus fibrations admit a Lagrangian section, then the $B$-field is unnecessary for the symmetry to work out; see for example Hitchin [24] or Polishchuk [34]. In the more general version involving a $B$-field, the identification of Hitchin fibers as dual abelian varieties comes about through a choice of trivialization of the restriction of the gerbe. Moreover, if there is a canonical coherent choice of trivialization of the gerbe in all fibers, the "gerby" duality (see Hausel-Thaddeus [20, p. 202] for its definition) follows from the usual one described here. This would be the case if, for example, the gerbe were known to be trivial.
3.2. The (stringy) E-polynomial. Let $M$ be a semiprojective variety (see [18]). This means that $M$ is quasi-projective and that (i) it carries an algebraic $\mathbb{C}^{*}$-action such that for any point $p$ in $M$, the limit of the $\mathbb{C}^{*}$-orbit $(t \cdot p)_{t \in \mathbb{C}^{*}}$ when $t$ goes to 0 exists in $M$, and (ii) the subvarieties of $M$ of fixed points under $\mathbb{C}^{*}$ are compact. Then by [18, Corollary 1.3.2] if $M$ is smooth, the (compactly supported) cohomology of $M$ is pure. Hence its E-polynomial is given by

$$
E(M)=\sum_{p, q=0}^{\operatorname{dim}(M)}(-1)^{p+q} h_{c}^{p, q}(M) u^{p} v^{q},
$$

where $h_{c}^{p, q}(M)=\operatorname{dim} H_{c}^{p, q}(M, \mathbb{C})$. Also, if $M$ has an action of a group $\Gamma$, we let $E(M)^{\Gamma}$ denote the $\Gamma$-invariant $E$-polynomial, i.e.,

$$
\begin{equation*}
E(M)^{\Gamma_{n}}=\sum_{p, q=0}^{\operatorname{dim}(M)}(-1)^{p+q} \operatorname{dim} H_{c}^{p, q}(M)^{\Gamma} u^{p} v^{q}, \tag{3.9}
\end{equation*}
$$

where $H_{c}^{p, q}(M)^{\Gamma} \subset H_{c}^{p, q}(M)$ is the $\Gamma$-invariant subspace.

The main motivational example for the definition of semiprojective varieties comes precisely from the moduli spaces of Higgs bundles. Indeed, they carry an algebraic $\mathbb{C}^{*}$ action, also in the strongly parabolic case, defined by

$$
\begin{equation*}
t \cdot(V, \varphi)=(V, t \varphi), t \in \mathbb{C}^{*} \tag{3.10}
\end{equation*}
$$

The following proposition describes two well-known fundamental properties of this action, which in particular show that $\mathcal{M}$ is semiprojective.

Proposition 3.7. The $\mathbb{C}^{*}$-action on $\mathcal{M}$ verifies the following properties.
(1) For any point $(V, \varphi) \in \mathcal{M}$, the limit $\lim _{t \rightarrow 0}(V, t \varphi)$ exists in $\mathcal{M}$ and is a fixed point.
(2) The subvarieties of $\mathcal{M}$ of fixed points are compact.

Therefore, $\mathcal{M}$ is semiprojective.
Proof. This is well known. The first item follows from the properness of the Hitchin map (3.1), just as in [37, Corollary 9.20]. Regarding the second item, the $\mathbb{C}^{*}$-fixed points are precisely the critical points of the real function $f: \mathcal{M} \rightarrow \mathbb{R}_{\geqslant 0}$ given by the $L^{2}$-norm of the Higgs field (see Proposition 3.3 of [13). By Theorem 4.1 (a) of [9], $f$ is proper, hence (2) also follows.

The moduli space $\mathcal{M} / \Gamma_{n}$ has orbifold singularities and, following Hausel-Thaddeus [20] (in turn based on Batyrev-Dais [5] and Batyrev [4]), we consider the stringy E-polynomial of $\mathcal{M} / \Gamma_{n}$, defined as

$$
\begin{equation*}
E_{\mathrm{st}}\left(\mathcal{M} / \Gamma_{n}\right)=\sum_{\gamma \in \Gamma_{n}} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)} \tag{3.11}
\end{equation*}
$$

where the various objects on the right hand side will be defined in the following. We note in passing that this stringy $E$-polynomial encodes the Chen-Ruan cohomology of $\mathcal{M} / \Gamma_{n}$ as an orbifold [11.

The subspace $\mathcal{M}^{\gamma} \subset \mathcal{M}$ is the locus pointwise fixed by $\gamma$. Since it is abelian, $\Gamma_{n}$ preserves $\mathcal{M}^{\gamma}$ and then $E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}$ is defined as in (3.9). The fermionic shift $F(\gamma)$ is defined as follows: given $p \in \mathcal{M}^{\gamma}$, the element $\gamma$ acts on the tangent space $T_{p} \mathcal{M}$ with eigenvalues $\lambda_{1}, \ldots, \lambda_{\operatorname{dim}(\mathcal{M})}$. Since $\Gamma_{n}$ is finite, these are roots of the unity, hence we can write $\lambda_{j}=e^{2 \pi i w_{j}}$, with $0 \leqslant w_{j}<1$ and $w_{j} \in \mathbb{Q}$. The fermionic shift is the number

$$
\begin{equation*}
F(p, \gamma)=\sum_{j=1}^{\operatorname{dim}(\mathcal{M})} w_{j} . \tag{3.12}
\end{equation*}
$$

Clearly it is constant along the connected component of $\mathcal{M}^{\gamma}$ containing $p$. In general the fermionic shift is just a rational number but, in our case, we can be much more precise. Recall that $\gamma$ acts by $\gamma \cdot(V, \varphi)=\left(V \otimes L_{\gamma}, \varphi \otimes \operatorname{Id}_{L_{\gamma}}\right)$, where $L_{\gamma}$ is the corresponding $n$-torsion line bundle. Recall that $\mathcal{N}$ denotes the moduli space of parabolic vector bundles with fixed determinant $\Lambda$ and the same parabolic structure as the one considered in $\mathcal{M}$. Then $\gamma$ acts by diffeomorphisms on $\mathcal{N}$, hence acts by symplectomorphisms on the cotangent bundle $T^{*} \mathcal{N}$, which is an open and dense subspace of $\mathcal{M}$, so $\gamma$ acts by symplectomorphisms on $\mathcal{M}$. It follows that for each eigenvalue $\lambda_{j}, \lambda_{j}^{-1}$ is also an eigenvalue. Since $\gamma$ acts trivially on the subspace $T_{p} \mathcal{M}^{\gamma} \subseteq T_{p} \mathcal{M}$, we conclude that

$$
\begin{equation*}
F(p, \gamma)=\sum_{l=1}^{\operatorname{dim}\left(N_{p} \mathcal{M}^{\gamma}\right) / 2}\left(w_{l}+1-w_{l}\right)=\frac{\operatorname{dim}\left(N_{p} \mathcal{M}^{\gamma}\right)}{2} . \tag{3.13}
\end{equation*}
$$

where $N_{p} \mathcal{M}^{\gamma} \subseteq T_{p} \mathcal{M}$ denotes the normal bundle to $\mathcal{M}^{\gamma}$ at $p$. We have already observed that, in general, $F(p, \gamma)$ only depends on the connected component of $\mathcal{M}^{\gamma}$ containing $p$. We shall in fact see in Section 6 that $\mathcal{M}^{\gamma}$ is non-connected, but we shall also conclude
directly (see (6.19) below) that the value of $F(p, \gamma)$ is really independent of $p$, thus independent of the component where it lies. That is the reason why we just wrote $F(\gamma)$ in the definition (3.11) of the stringy $E$-polynomial.
Remark 3.8. Note that $F(e)=0$ where $e$ is the trivial element of $\Gamma_{n}$. Thus the stringy $E$-polynomial of a smooth variety coincides with the usual one. In particular we have that $E(\mathcal{M})=E_{\mathrm{st}}(\mathcal{M})$.

We conclude this section by pointing out that the $E$-polynomials are independent of the weights $\boldsymbol{\alpha}$, as long as these are generic. This will be useful in our later calculations (specifically, in the proof of Proposition 4.5) since it allows us to make simplifying assumptions on the weights. Everything follows from the work of Thaddeus [39, who studied how the moduli space of parabolic Higgs bundles changes under wall crossing of the parabolic weights. It is immediate from his description that the Hodge numbers of the moduli space are unchanged under wall crossing. We shall need a $\Gamma_{n}$-equivariant version of this result. This also follows from Thaddeus' description, which we now briefly recall.

Let $\boldsymbol{\alpha}$ belong to only one wall in the space of parabolic weights and consider weights $\boldsymbol{\alpha}^{-}$and $\boldsymbol{\alpha}^{+}$in the two adjacent chambers. For brevity write $\mathcal{M}^{ \pm}$for either of the moduli spaces $\mathcal{M}_{\Lambda}^{\alpha^{ \pm}}(\mathrm{SL}(n, \mathbb{C}))$ and $\mathcal{M}_{d}^{\alpha^{ \pm}}(\mathrm{GL}(n, \mathbb{C}))$ (everything in this section applies to both of these). There are flip loci $\mathcal{S}^{ \pm} \subset \mathcal{M}^{ \pm}$which correspond to those parabolic Higgs bundles which are $\boldsymbol{\alpha}_{ \pm}$-stable and $\boldsymbol{\alpha}_{\mp}$-unstable. In the following, write $\mathbf{V}=(V, \varphi)$ for a parabolic Higgs bundle. Points of $\mathcal{S}^{-}$correspond to parabolic Higgs bundles $\mathbf{V}$ which are non-split extensions

$$
\begin{equation*}
0 \rightarrow \mathbf{V}^{+} \rightarrow \mathbf{V} \rightarrow \mathbf{V}^{-} \rightarrow 0 \tag{3.14}
\end{equation*}
$$

of parabolic Higgs bundles, where $\mathbf{V}^{ \pm}$are stable with respect to the parabolic weights induced by $\boldsymbol{\alpha}^{ \pm}$. There is an analogous description of $\mathcal{S}^{+}$. Thus there is a natural identification

$$
g: \mathcal{M}^{-} \backslash \mathcal{S}^{-} \stackrel{\cong}{\rightrightarrows} \mathcal{M}^{+} \backslash \mathcal{S}^{+}
$$

Denote by $\pi^{ \pm}: \tilde{\mathcal{M}}^{ \pm} \rightarrow \mathcal{M}^{ \pm}$the blow-ups of $\mathcal{M}^{ \pm}$along $\mathcal{S}^{ \pm}$and by $\mathcal{E}^{ \pm} \subset \tilde{\mathcal{M}}^{ \pm}$the exceptional divisors. Thaddeus [39, (6.2)] proves that there is an isomorphism

$$
\begin{equation*}
\tilde{g}: \tilde{\mathcal{M}}^{-} \xrightarrow{\cong} \tilde{\mathcal{M}}^{+} \tag{3.15}
\end{equation*}
$$

which restricts to an isomorphism $\mathcal{E}^{-} \xrightarrow{\cong} \mathcal{E}^{+}$of the exceptional divisors and coincides with $g$ on their complement. It is a standard fact about blow-ups that the cohomology groups of $\mathcal{M}^{ \pm}$inject into the cohomology groups of $\tilde{\mathcal{M}}^{ \pm}$and from (3.15) it follows that $\tilde{g}$ induces isomorphisms

$$
\begin{equation*}
H_{c}^{p, q}\left(\mathcal{M}^{-}\right) \cong H_{c}^{p, q}\left(\mathcal{M}^{+}\right) \tag{3.16}
\end{equation*}
$$

considering these cohomology groups as subspaces of $H_{c}^{p, q}\left(\tilde{\mathcal{M}}^{ \pm}\right)$. Thus, for generic $\boldsymbol{\alpha}$, the $E$-polynomials of $\mathcal{M}_{d}^{\alpha}(\mathrm{GL}(n, \mathbb{C}))$ and $\mathcal{M}_{\Lambda}^{\alpha}(\mathrm{SL}(n, \mathbb{C}))$ are independent of $\boldsymbol{\alpha}$.

In view of what we have said so far, it is now easy to prove the following.
Proposition 3.9. Let $\Gamma_{n}=\operatorname{Jac}_{[n]}(X)$ act on the moduli space of parabolic Higgs bundles by the action defined in (2.4). The isomorphism $\tilde{g}$ of (3.15) is equivariant with respect to this action. Consequently, the isomorphism (3.16) is also $\Gamma_{n}$-equivariant.
Proof. The basic observation is that the action of $\Gamma_{n}$ preserves $\mathcal{S}^{ \pm} \subset \mathcal{M}^{ \pm}$; this follows from the description of $\mathcal{S}^{-}$(and the analogous description of $\mathcal{S}^{+}$) as corresponding to extensions of the form (3.14). Hence the $\Gamma_{n}$-actions lift to the blow-ups $\tilde{\mathcal{M}}^{ \pm}$(as follows from the universal property of the blow-up, Hartshorne [17, Cor. II.7.15]). Moreover, the
restriction of $\tilde{g}$ to the open dense subset $\tilde{\mathcal{M}}^{-} \backslash \mathcal{E}^{-} \subset \tilde{\mathcal{M}}^{-}$is just $g$, which is $\Gamma_{n}$-equivariant by our initial basic observation. It follows that $\tilde{g}$ is $\Gamma_{n}$-equivariant as claimed.

Corollary 3.10. Assume that $\boldsymbol{\alpha}$ is generic and let $\mathcal{M}^{\alpha}$ denote either $\mathcal{M}_{d}^{\alpha}(\operatorname{GL}(n, \mathbb{C}))$ or $\mathcal{M}_{\Lambda}^{\alpha}(\mathrm{SL}(n, \mathbb{C}))$. Then the compactly supported Dolbeault cohomology of $\mathcal{M}^{\alpha}$ is independent of $\boldsymbol{\alpha}$ as a $\Gamma_{n}$-module. Thus the E-polynomial $E\left(\mathcal{M}^{\alpha}\right)$ and the $\Gamma_{n}$-invariant E-polynomial $E\left(\mathcal{M}^{\alpha}\right)^{\Gamma_{n}}$ are both independent of $\boldsymbol{\alpha}$.

Remark 3.11. We shall see that the stringy E-polynomial of the moduli space of parabolic $\operatorname{PGL}(n, \mathbb{C})$-Higgs bundles is also independent of $\boldsymbol{\alpha}$. Indeed, it will follow from the description given in Theorem 6.3 below that for any $e \neq \gamma \in \Gamma_{n}$, the parabolic Higgs bundles in the fixed locus $\left(\mathcal{M}^{\alpha}\right)^{\gamma} \subset \mathcal{M}^{\alpha}$ are $\boldsymbol{\alpha}$-semistable for any value of $\boldsymbol{\alpha}$. In other words $\left(\mathcal{M}^{\alpha}\right)^{\gamma}$ does not intersect the flip locus $\mathcal{S}^{ \pm}$and thus $\tilde{g}$ from (3.15) restricts to a $\Gamma_{n}$-equivariant isomorphism. Thus all the terms in the definition (3.11) of the stringy $E$-polynomial are independent of $\boldsymbol{\alpha}$.
3.3. Topological mirror symmetry and the main result. The topological mirror symmetry conjecture of Hausel-Thaddeus says that the stringy E-polynomials of the mirror partners $\mathcal{M}$ and $\mathcal{M} / \Gamma_{n}$ should agree. Since the SYZ mirror symmetry statement is really about the de Rham moduli spaces, rather than the Dolbeault moduli spaces, so is the topological mirror symmetry conjecture (see Remark 3.1). On the other hand, it is the rich algebraic geometry of the Higgs bundle moduli spaces and, in particular, the fact that it carries a $\mathbb{C}^{*}$-action which allows Hausel and Thaddeus [20] to prove the equality of the $E$-polynomials in the non-parabolic case. This suffices because they also prove that the de Rham and Dolbeault moduli spaces have the same E-polynomials. The proof of this latter result uses that the Dolbeault and de Rham moduli spaces live in a family, the Hodge moduli space, which parametrizes so-called $\lambda$-connections. This moduli space fibers over the affine line $\mathbb{C}$ with fibers away from zero all isomorphic to the de Rham moduli space and degenerating to the Dolbeault moduli space over zero. Parabolic $\lambda$ connections and the corresponding moduli spaces were constructed and studied by AlfayaGómez [1] and their results provide the necessary input for applying the arguments of Hausel-Thaddeus [20, Sec. 6] (cf. Hausel-Rodriguez-Villegas [18, Cor. 1.3.3]) directly in the parabolic situation. Thus the parabolic de Rham moduli spaces have the same $E$ polynomials as the moduli spaces of parabolic Higgs bundles, and we can exclusively work with the latter for the remainder of the paper.

The topological mirror symmetry conjecture can now be stated in terms of the Higgs bundle moduli spaces as follows.

Conjecture 3.12 (Hausel-Thaddeus [19, 20]). For any rank n, any line bundle $\Lambda$ and any system of generic weights $\boldsymbol{\alpha}$, the equality of $E$-polynomials

$$
\begin{equation*}
E\left(\mathcal{M}_{\Lambda}^{\alpha}\right)=E_{\mathrm{st}}\left(\mathcal{M}_{\Lambda}^{\alpha} / \Gamma_{n}\right) \tag{3.17}
\end{equation*}
$$

holds.
Our main result states that this is true for $n=2,3$.
Theorem 3.13. If $n=2,3$, then Conjecture 3.12 holds.
Proof. We follow the strategy of [20] which we now explain. From the definition of the stringy $E$-polynomial (3.11) of $\mathcal{M} / \Gamma_{n}$ and from Remark 3.8, we have that

$$
E_{\mathrm{st}}\left(\mathcal{M} / \Gamma_{n}\right)=E(\mathcal{M})^{\Gamma_{n}}+\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}
$$

On the other hand, let $E(\mathcal{M})^{\mathrm{var}}$ denote the variant part of $E(\mathcal{M})$ in Hausel and Thaddeus' terminology. It is defined analogously to $E(\mathcal{M})$ but the coefficients are given by subtracting the dimensions of the $\Gamma_{n}$-invariant subspaces, i.e.,

$$
E(\mathcal{M})=E(\mathcal{M})^{\Gamma_{n}}+E(\mathcal{M})^{\mathrm{var}}
$$

Hence (3.17) is equivalent to

$$
\begin{equation*}
E(\mathcal{M})^{\mathrm{var}}=\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)} \tag{3.18}
\end{equation*}
$$

Now, Theorems 3.14 and 3.15 below imply that (3.18) holds for any $n=2,3$, proving Theorem 3.13,

Thus the following two theorems complete the proof of Theorem 3.13, Here $\mathcal{F}_{(1,1, \ldots, 1)}$ denotes the subspace of $\mathcal{M}$ consisting of subvarieties of fixed points of the $\mathbb{C}^{*}$-action (3.10) of type $(1,1, \ldots, 1)$, to be properly defined in the following section (see in particular (4.1)), and $E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}$ is the variant part of the corresponding $E$-polynomial.

Theorem 3.14. Let $n=2,3$. For any system of generic weights $\boldsymbol{\alpha}$, and any line bundle $\Lambda$, we have $E(\mathcal{M})^{\mathrm{var}}=(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}$.
Proof. For $n=2$, this follows from comparing Holla [25, Theorem 5.23] and Nitsure [33, Proposition 3.11] and using the argument of Atiyah-Bott [3, Prop. 9.7]; see also [33, Remark 3.11] and [13, Remark 10.1]. For $n=3$, it follows from [13, Theorem 12.22]; see also Remarks 12.17 and 12.19 of loc. cit..

We shall say that a result holds for small weights if there is an $\epsilon>0$ such that the result holds for any system of weights $\boldsymbol{\alpha}$ with $\alpha_{i}(p)<\epsilon$ for all $i$ and $p$.
Theorem 3.15. For any $n$ prime, any system of small generic weights $\boldsymbol{\alpha}$, and any line bundle $\Lambda$, we have

$$
\begin{equation*}
(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}=\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)} \tag{3.19}
\end{equation*}
$$

and both sides are equal to

$$
\begin{equation*}
\frac{n^{2 g}-1}{n}(n!)^{|D|}(u v)^{\left(n^{2}-1\right)(g-1)+|D| n(n-1) / 2}((1-u)(1-v))^{(n-1)(g-1)} . \tag{3.20}
\end{equation*}
$$

Moreover, if $n=2,3$, the result holds for any system of generic weights.
Remark 3.16. When $n=2$, the polynomial (3.20) is equivalent to the one which appears in [19], the difference in sign being due to different conventions.

Remark 3.17. We shall conclude directly in Section 6 that the right-hand side of (3.19) is independent of the generic weights (see also Remark 3.11), so that the assumption on small weights is only needed to compute $E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}$. Specifically, it is used in the proof of Proposition 4.5 below. Now, it follows from Corollary 3.10 that the polynomial $E(\mathcal{M})^{\mathrm{var}}$ is independent of the generic weights, so if Theorem 3.14 were known to be true for any prime $n$, then we could remove this small weights assumption from Theorem 3.15, The only obstacle for having a proof of Theorem 3.13 for any $n$ prime is thus the fact that Theorem 3.14 is not known to hold for such $n$.

The remaining part of the paper will be dedicated to the proof of Theorem 3.15, which follows from Proposition 4.5, Corollary 4.7 and Proposition 6.7 below. Again we follow the arguments of [20]. We shall prove that both $(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\text {var }}$ and $\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}$ are equal to the given polynomial (in the former case for
small weights if $n>3$ ). The proofs of these equalities are completely independent of each other. The case of $(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\text {var }}$ will be treated in Section 4, while the case of $\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}$ is going to be dealt with in Section 6. Section 5 is an independent section, containing some results on Prym varieties of unramified covers, which are needed in Section 6 .

## 4. The polynomial $(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}$

The $\mathbb{C}^{*}$-action (3.10) on the moduli space $\mathcal{M}$ is a fundamental tool on the study of its geometry and topology. In particular the cohomology of $\mathcal{M}$ is completely determined by the cohomology of the subvarieties of fixed points, hence so is the $E$-polynomial of $\mathcal{M}$. Here we aim to compute the $E$-polynomial of a certain subspace of the fixed point loci of the $\mathbb{C}^{*}$-action, relevant for Theorem 3.15. In the next subsections, we describe these fixed point locus.
4.1. The fixed points of the $\mathbb{C}^{*}$-action. Here we shall consider the fixed point subvarieties of the $\mathbb{C}^{*}$-action (3.10). From Proposition 3.7 we know that these are compact, but now we need a more explicit description of the fixed points. This is provided by the following result due to Simpson (see [36, Theorem 8]).

Proposition 4.1. A stable parabolic $\operatorname{SL}(n, \mathbb{C})$-Higgs bundle $(V, \varphi) \in \mathcal{M}$ is a fixed point under $\mathbb{C}^{*}$ if and only if either
(1) $\varphi \equiv 0$ or
(2) $V$ admits a decomposition $V \cong \bigoplus_{j=1}^{l} V_{j}$ such that the following hold:

- the subbundles $V_{j}$ are parabolic and the decomposition $V \cong \bigoplus_{j=1}^{l} V_{j}$ is compatible with the parabolic structure, i.e., at every point $p \in D$, every subspace $V_{p, i}$ is a direct sum of fibers at $p$ of certain subbundles $V_{j}$.
- the Higgs field splits as $\varphi=\sum_{j=1}^{l} \varphi_{j}$, with $\varphi_{j}: V_{j} \rightarrow V_{j+1} \otimes K(D)$ non-zero for all $j=1, \ldots, l-1$, and $\varphi_{l} \equiv 0$.
A parabolic Higgs bundle of the kind described in the preceding proposition is called a Hodge bundle. Note that we can include the ones of the form ( $V, 0$ ) in point (2) by taking $l=1$, however it will be convenient for us to distinguish the two kinds of fixed points notationally.

Definition 4.2. A fixed point with non-vanishing Higgs field is said to be of type $\left(n_{1}, n_{2}, \ldots, n_{l}\right)$, with $\sum n_{j}=n$, if $\operatorname{rk}\left(V_{j}\right)=n_{j}$, for all $j$. Denote by $\mathcal{F}_{\left(n_{1}, n_{2}, \ldots, n_{l}\right)}$ the union of the subvarieties of $\mathcal{M}$ of all fixed points of type $\left(n_{1}, n_{2}, \ldots, n_{l}\right)$.

As is well known, it follows from Bialynicki-Birula stratification associated to the $\mathbb{C}^{*}$ action that the cohomology of $\mathcal{M}$ is determined by the cohomology of all fixed point subvarieties of the $\mathbb{C}^{*}$-action. Indeed, the $\mathbb{C}^{*}$-flows gives rise to Zariski locally trivial affine bundles, with fibre $\mathbb{C}^{\operatorname{dim}(\mathcal{M}) / 2}$, over the disjoint union of all $\mathcal{F}_{\left(n_{1}, n_{2}, \ldots, n_{l}\right)}$ together with $\mathcal{N}$. This follows by Proposition 3.7 (1), and the projection of these affine bundles is just taking the limit of the flow when $t$ goes to 0 . Since the $E$-polynomial is additive with respect to disjoint unions and multiplicative with respect to locally trivial fibrations in the Zariski topology, we consequently have that

$$
\begin{equation*}
E(\mathcal{M})=(u v)^{\operatorname{dim}(\mathcal{M}) / 2}\left(E(\mathcal{N})+\sum_{\left(n_{1}, n_{2}, \ldots, n_{l}\right)} E\left(\mathcal{F}_{\left(n_{1}, n_{2}, \ldots, n_{l}\right)}\right)\right) \tag{4.1}
\end{equation*}
$$

All $\mathcal{F}_{\left(n_{1}, n_{2}, \ldots, n_{l}\right)}$ and $\mathcal{N}$ are smooth and projective so we can consider their usual $E$ polynomials.

According to Theorem [3.15, the relevant subvarieties to be considered are the ones corresponding to type $(1,1, \ldots, 1)$, that is $\mathcal{F}_{(1,1, \ldots, 1)}$.
4.2. The subvarieties $\mathcal{F}_{(1,1, \ldots, 1)}$. Let $n$ be a prime number. Our next task is to obtain a geometric description of the subspace $\mathcal{F}_{(1,1, \ldots, 1)}$. If $(V, \varphi)$ represents a fixed point of the $\mathbb{C}^{*}$-action of type $(1,1, \ldots, 1)$ then

$$
\begin{equation*}
V=\bigoplus_{j=1}^{n} L_{j} \quad \text { and } \quad \varphi=\sum_{j=1}^{n-1} \varphi_{j}, \quad \varphi_{j}: L_{j} \rightarrow L_{j+1} \otimes K(D), \varphi_{n} \equiv 0 \tag{4.2}
\end{equation*}
$$

Since in $\mathcal{M}$ we always have fixed determinant $\Lambda$, then

$$
\begin{equation*}
\prod_{L_{j}} \simeq \Lambda . \tag{4.3}
\end{equation*}
$$

The subspace $\mathcal{F}_{(1,1, \ldots, 1)}$ is decomposed into connected components which can be labeled by the topological data coming from decomposition (4.2), namely the degrees of the bundles $L_{j}$ and the way the weights are distributed among them at each point of $D$. Actually, instead of using the degrees of the bundles $L_{j}$, we shall opt for a slight variation of this.

Over each $p \in D$, we have the corresponding parabolic structure

$$
\begin{equation*}
V_{p}=V_{p, 1} \supsetneq V_{p, 2} \supsetneq \cdots \supsetneq V_{p, n} \supsetneq\{0\}, \quad 0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1 . \tag{4.4}
\end{equation*}
$$

By Proposition 4.1, each $L_{j}$ is a parabolic subbundle of $V$ and the decomposition (4.2) is compatible with the parabolic structure (4.4). The filtration of the fibre $L_{j, p}$ of $L_{j}$ at $p$ is of course trivial

$$
\begin{equation*}
L_{j, p} \supsetneq\{0\}, \tag{4.5}
\end{equation*}
$$

and the corresponding weight $\beta_{j}(p)$ assigned to $L_{j, p}$ is $\beta_{j}(p)=\alpha_{i}(p)$ where $i$ is such that $L_{j, p} \subseteq V_{p, i}$ but $L_{j, p} \nsubseteq V_{p, i+1}$; this is precisely the condition coming from (2.2). Since there are $n$ line subbundles and the filtration (4.4) has length $n$, we see that (4.4) is determined by a distribution of the weights at $p$ among the fibers of the line subbundles $L_{j}$ at $p$. Precisely, $V_{p, n}=L_{j, p}$ where $j$ is such that $\beta_{j}(p)=\alpha_{n}(p)$ and, for $i<n$, $V_{p, i}=V_{p, i+1} \oplus L_{j^{\prime}, p}$ with $j^{\prime}$ such that $\beta_{j^{\prime}}(p)=\alpha_{i}(p)$. Such distribution of the $n$ weights at $p$ is provided by a permutation of the set $\{1, \ldots, n\}$, so by an element $\varpi_{n}(p)$ of the symmetric group $S_{n}$. Write such permutation by a word

$$
\varpi_{n}(p)=a_{1}(p) a_{2}(p) \ldots a_{n}(p) \in S_{n}
$$

with $a_{j}(p) \in\{1, \ldots, n\}$, where this means that we assign the weight $\alpha_{a_{j}(p)}(p)$ to the fibre $L_{j, p}$. The conclusion is that the parabolic structure on $V=\bigoplus_{j=1}^{n} L_{j}$ is determined by an element

$$
\begin{equation*}
\varpi_{n}=\left(\varpi_{n}\left(p_{1}\right), \ldots, \varpi_{n}\left(p_{|D|}\right)\right) \in S_{n}^{|D|} . \tag{4.6}
\end{equation*}
$$

Now we have to see how the Higgs field comes into play. It is given by (4.2), so $\varphi_{j} \in H^{0}\left(X, \operatorname{SParHom}\left(L_{j}, L_{j+1}\right) \otimes K(D)\right)$ for every $j$. The residue of $\varphi$ at $p \in D$ is given, according to the decomposition (4.2) of $V$, by

$$
\varphi_{p}=\left(\begin{array}{ccccc}
0 & 0 & \ldots & 0 & 0 \\
\varphi_{1, p} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \varphi_{n-1, p} & 0
\end{array}\right)
$$

Suppose $\varpi_{n}(p)=a_{1}(p) a_{2}(p) \ldots a_{n}(p)$. Since $\varphi$ is strongly parabolic, it follows from (4.5) that if $a_{j}(p)>a_{j+1}(p)$ then $\varphi_{j, p}=0$. Thus

$$
a_{j}(p)>a_{j+1}(p) \Longrightarrow \varphi_{j} \in H^{0}\left(X, \operatorname{Hom}\left(L_{j}, L_{j+1}\right) \otimes K(D-p)\right)
$$

For each $j=1, \ldots, n-1$, define the subdivisor of $D$

$$
S_{j}\left(\varpi_{n}\right)=\left\{p \in D \mid a_{j}(p)>a_{j+1}(p)\right\} \subseteq D
$$

so that

$$
\varphi_{j} \in H^{0}\left(X, \operatorname{Hom}\left(L_{j}, L_{j+1}\right) \otimes K\left(D-S_{j}\left(\varpi_{n}\right)\right)\right)
$$

Let

$$
M_{j}=L_{j}^{-1} L_{j+1} K\left(D-S_{j}\left(\varpi_{n}\right)\right)
$$

and write

$$
\begin{equation*}
m_{j}=\operatorname{deg}\left(M_{j}\right)=-d_{j}+d_{j+1}+2 g-2+|D|-s_{j}\left(\varpi_{n}\right) \geqslant 0 \tag{4.7}
\end{equation*}
$$

with $d_{j}=\operatorname{deg}\left(L_{j}\right)$ and $s_{j}\left(\varpi_{n}\right)=\left|S_{j}\left(\varpi_{n}\right)\right|$, the cardinal of $S_{j}\left(\varpi_{n}\right)$. By (4.3),

$$
\begin{equation*}
\prod_{j=1}^{n-1} M_{j}^{j} \cong L_{n}^{n} \Lambda^{-1} K^{\frac{n(n-1)}{2}}\left(\frac{n(n-1)}{2} D-\sum_{j=1}^{n-1} j S_{j}\left(\varpi_{n}\right)\right) \tag{4.8}
\end{equation*}
$$

and this implies

$$
\left\{\begin{array}{lll}
d+\sum_{j=1}^{n-1} j\left(m_{j}+s_{j}\left(\varpi_{n}\right)\right) \equiv 0(\bmod n), & \text { if } & n \geqslant 3  \tag{4.9}\\
d+m_{1}+s_{1}\left(\varpi_{2}\right)-|D| \equiv 0(\bmod 2), & \text { if } & n=2
\end{array}\right.
$$

Clearly the collection $\left(m_{j}\right)_{j}$ determines the collection $\left(d_{j}\right)_{j}$ and vice-versa through (4.7) and (4.8).
The proper $\varphi$-invariant subbundles of $V$ are the ones of the form $V_{l}=\bigoplus_{j=l}^{n} L_{j}$, for $2 \leqslant l \leqslant n$. The stability condition $\operatorname{par} \mu\left(V_{l}\right)<\operatorname{par} \mu(V)$ (cf. Definition 2.11) for the subbundle $V_{l}$ reads as

$$
\begin{align*}
(n-l+1) \sum_{j=1}^{l-1} j m_{j}+(l-1) \sum_{j=l}^{n-1}(n-j) m_{j} & <\sum_{p \in D}\left(\sum_{i=1}^{n}(n-l+1) \alpha_{i}(p)-n \sum_{j=l}^{n} \alpha_{a_{j}(p)}(p)\right)+  \tag{4.10}\\
& +(g-1+|D| / 2) n(n-l+1)(l-1)- \\
& -(n-l+1) \sum_{j=1}^{l-1} j s_{j}\left(\varpi_{n}\right)-(l-1) \sum_{j=l}^{n-1}(n-j) s_{j}\left(\varpi_{n}\right) .
\end{align*}
$$

Given $\varpi_{n}$ as in (4.6) and $m_{1}, \ldots, m_{n-1}$ non-negative integers such that (4.9) and (4.10) hold, denote by $\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)$ be the subspace of $\mathcal{F}_{(1,1, \ldots, 1)}$ determined by the given numerical/topological data. So we can write the decomposition of $\mathcal{F}_{(1,1, \ldots, 1)}$ as

$$
\begin{equation*}
\mathcal{F}_{(1,1, \ldots, 1)}=\bigsqcup_{\varpi_{n} \in S_{n}^{|D|}} \bigsqcup_{\substack{m_{1}, \ldots, m_{n},-1 \geq 0 \\ \text { such that } \frac{4.9}{4.9}, 4100 \\ \text { hold }}} \mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right), \tag{4.11}
\end{equation*}
$$

and, from what we have done so far, the following is clear.
Proposition 4.3. Let $\varpi_{n} \in S_{n}^{|D|}$ as in (4.6) and $m_{1}, \ldots, m_{n-1}$ non-negative integers verifying (4.9) and (4.10) for every $l=2, \ldots, n$. Then the critical subvariety $\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)$ is given by the pull-back diagram

where:

- the top map is $(V, \varphi)=\left(\bigoplus_{j} L_{j}, \sum_{j} \varphi_{j}\right) \mapsto L_{n}$;
- $d_{n}=\frac{1}{n} \sum_{j=1}^{n-1} j\left(m_{j}+s_{j}\left(\varpi_{n}\right)\right)-(n-1)(g-1+|D| / 2)$;
- the vertical map on the left is given by $(V, \varphi)=\left(\bigoplus_{j} L_{j}, \sum_{j} \varphi_{j}\right) \mapsto\left(\operatorname{div}\left(\varphi_{1}\right), \ldots, \operatorname{div}\left(\varphi_{n-1}\right)\right)$;
- the map on the bottom is $\left(D_{1}, \ldots, D_{n-1}\right) \mapsto \mathcal{O}_{X}\left(\sum_{j} j D_{j}\right)$;
- the vertical map on the right is $L_{n} \mapsto L_{n}^{n} \Lambda^{-1} K^{\frac{n(n-1)}{2}}\left(\frac{n(n-1)}{2} D-\sum_{j=1}^{n-1} j S_{j}\left(\varpi_{n}\right)\right)$.
4.3. The $E$-polynomial of the variant part. The proof of the next result uses the description $\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)$ given in Proposition 4.3, It can be found essentially in [21, Theorem 7.6 (iv)], [15, Proposition 3.11] and [20, Proposition 10.1]. Again it is essential that $n$ is prime.

Recall that the group $\Gamma_{n}$ acts on $\mathcal{M}$ by (2.4). This action clearly preserves each component $\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)$ of $\mathcal{F}_{(1,1, \ldots, 1)}$.
Proposition 4.4. Let $n$ be prime. The variant part of the cohomology of $\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)$ is non-trivial only in degree $m_{1}+\cdots+m_{n-1}$. More precisely,

$$
H^{*}\left(\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right), \mathbb{C}\right)^{\mathrm{var}} \cong \bigoplus_{\gamma \in \Gamma_{n} \backslash\{e\}} \bigotimes_{j=1}^{n-1} \Lambda^{m_{j}} H^{1}\left(X, L_{\gamma}^{j}\right),
$$

where $H^{1}\left(X, L_{\gamma}^{j}\right)$ denotes twisted cohomology with values in the local system $L_{\gamma}^{j}$, and $L_{\gamma}$ is the flat line bundle corresponding to $\gamma$.

We are now in position to determine the left-hand side of (3.19).
Proposition 4.5. For any $n$ prime, and any generic system of small weights, the following holds:
$(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}=\frac{n^{2 g}-1}{n}(n!)^{|D|}(u v)^{\left(n^{2}-1\right)(g-1)+|D| n(n-1) / 2}((1-u)(1-v))^{(n-1)(g-1)}$.
Proof. By (4.11),

$$
\begin{equation*}
E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}=\sum_{\varpi_{n} \in S_{n}^{|D|}} \sum_{\substack{m_{1}, \ldots, m_{n}-1 \\ \text { such that }}} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)\right)^{\mathrm{var}}, \tag{4.13}
\end{equation*}
$$

and then we must multiply it by the factor $(u v)^{\operatorname{dim}(\mathcal{M}) / 2}$.
For any non-trivial $\gamma \in \Gamma_{n}$ and any $j, \operatorname{dim} H^{1}\left(X, L_{\gamma}^{j}\right)=g-1$, i.e., $\operatorname{dim} H^{1,0}\left(X, L_{\gamma}^{j}\right)=$ $g-1$. Thus by Proposition 4.4, we find that
$E\left(\mathcal{F}_{(1,1, \ldots, 1)}\left(\varpi_{n}, m_{1}, \ldots, m_{n-1}\right)\right)^{\operatorname{var}}(u, v)=\left(n^{2 g}-1\right) \prod_{j=1}^{n-1} \sum_{\substack{p+q=m_{j} \\ 0 \leqslant p, q \leqslant g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q} u^{p} v^{q}$.
We need to sum this expression over all $k$-tuples of permutations $\varpi_{n}$ and over all nonnegative integers $m_{j}$ such that (4.9) and (4.10) hold.

Regarding the summation over the $m_{j}$, note that the right hand side is zero whenever there is an $m_{j}>2 g-2$.

Since the system of the (generic) weights is small, the summand

$$
\sum_{p \in D}\left(\sum_{i=1}^{n}(n-l+1) \alpha_{i}(p)-n \sum_{j=l}^{n} \alpha_{a_{j}(p)}(p)\right)
$$

in (4.10) is very close to zero as well. From this, and using the fact that $s_{j}\left(\varpi_{n}\right) \leqslant|D|$ for all $j$, one shows that (4.10) holds for $m_{1}=\cdots=m_{n-1}=2 g-2$, hence holds for any choice of $m_{j}$ between 0 and $2 g-2$ for every $j$. Therefore we can sum over
all $0 \leqslant m_{1}, \ldots, m_{n-1} \leqslant 2 g-2$ subject to condition (4.9). This is done by taking $\xi=\exp (2 \pi i / n)$ and since, for a given integer $\nu \in \mathbb{Z}$, the sum $\sum_{l=0}^{n-1} \xi^{l \nu}$ equals $n$ if $\nu \equiv 0$ $(\bmod n)$ and zero otherwise, we have that $E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}(u, v)$ in (4.13) equals, if $n \geqslant 3$,

$$
\begin{aligned}
& \frac{n^{2 g}-1}{n} \sum_{\varpi_{n} \in S_{n}^{|D|}} \sum_{m_{1}, \ldots, m_{n-1}=0}^{2 g-2} \sum_{l=0}^{n-1} \xi^{l} \sum_{j=1}^{n-1} j\left(m_{j}+s_{j}\left(\varpi_{n}\right)\right) \\
= & \prod_{j=1}^{n-1} \sum_{\substack{p+q=m_{j} \\
0 \leqslant p, q \leq g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q} u^{p} v^{q} \\
= & \frac{n^{2 g}-1}{n} \sum_{\varpi_{n} \in S_{n}^{|D|}} \sum_{l=0}^{n-1} \xi^{l \sum_{j=1}^{n-1} j s_{j}\left(\varpi_{n}\right)} \prod_{j=1}^{n-1} \sum_{m_{j}=0}^{2 g-2} \sum_{\substack{p+q=m_{j} \\
0 \leqslant p, q \leqslant g-1}}(-1)^{p+q}\binom{g-1}{p}\binom{g-1}{q} u^{p} v^{q} \xi^{j l m_{j}} \\
= & \frac{n^{2 g}-1}{n}(n!)^{|D|}((1-u)(1-v))^{l(n-1)(g-1)}+\frac{n^{2 g}-1}{n}\left(\frac{\left(1-u^{n}\right)\left(1-v^{n}\right)}{(1-u)(1-v)}\right)^{g-1}(n \mathcal{S}(n, d)-n!)^{|D|},
\end{aligned}
$$

where in the last equality we used the fact that $n$ is prime, and where

$$
\mathcal{S}(n, d)=\#\left\{\varpi_{n}(p) \in S_{n} \mid \sum_{j=1}^{n-1} j s_{j}\left(\varpi_{n}(p)\right) \equiv 0 \quad(\bmod n)\right\}
$$

For $n=2$ we perform the precise same computation, except that we use the expression corresponding to $n=2$ in (4.9), yielding

$$
\begin{aligned}
E\left(\mathcal{F}_{(1,1)}\right)^{\mathrm{var}} & =2^{|D|-1}\left(2^{2 g}-1\right)(u v)^{3 g-3+|D|}((1-u)(1-v))^{g-1} \\
& +\frac{2^{2 g}-1}{2}((1+u)(1+v))^{g-1}(2 \mathcal{S}(2, d)-2)^{|D|}
\end{aligned}
$$

where $\mathcal{S}(2, d)=\#\left\{\varpi_{2}(p) \in S_{2}| | D \mid+s_{1}\left(\varpi_{2}(p)\right) \equiv 0(\bmod 2)\right\}$.
It is clear that that the values of both $\mathcal{S}(n, d)$ and $\mathcal{S}(2, d)$ are independent of $p \in D$. It is also clear that $\mathcal{S}(2, d)=1$. Actually by Lemma 4.6 below, we have $\mathcal{S}(n, d)=(n-1)$ !, hence, for any $n \geqslant 2$ prime, $(u v)^{\operatorname{dim}(\mathcal{M}) / 2} E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\operatorname{var}}(u, v)$ equals

$$
\frac{n^{2 g}-1}{n}(n!)^{|D|}(u v)^{\left(n^{2}-1\right)(g-1)+|D| n(n-1) / 2}((1-u)(1-v))^{(n-1)(g-1)},
$$

completing the proof.
The next lemma completes the proof of Proposition 4.5.
Lemma 4.6. For any $n \geqslant 2$ and $d, \mathcal{S}(n, d)=(n-1)$ !.
Proof. This is a purely combinatorial proof. Since the number $\mathcal{S}(n, d)$ is obviously independent of $p \in D$, we will remove it from the notation. Any permutation $\varpi_{n} \in S_{n}$ is obtained from a permutation $\varpi_{n-1} \in S_{n-1}$ by inserting $n$ in the appropriate position. Conversely, any $\varpi_{n-1}=a_{1} a_{2} \ldots a_{n-1} \in S_{n-1}$ produces $n$ distinct permutations in $S_{n}$, by inserting $n$ in $\varpi_{n-1}$ in each one of the possible $j$ positions of $\varpi_{n-1}$, where $j \in\{0, \ldots, n-1\}$. Write $\varpi_{n-1}(j)$ for such permutation in $S_{n}$, so that

$$
\begin{aligned}
\varpi_{n-1}(0) & =n a_{1} \ldots a_{n-1}, \\
\varpi_{n-1}(j) & =a_{1} \ldots a_{j} n a_{j+1} \ldots a_{n-1} \quad 1 \leqslant j \leqslant n-2, \\
\varpi_{n-1}(n-1) & =a_{1} \ldots a_{n-1} n .
\end{aligned}
$$

Fix any $\varpi_{n-1} \in S_{n-1}$. Let $A$ be the ordered set of the indexes $i$ between 1 and $n-2$ where $s_{i}\left(\varpi_{n-1}\right)=1$. In other words,

$$
A=\left\{i_{1}, \ldots, i_{s} \mid i_{\ell}<i_{\ell+1}, a_{i_{\ell}}>a_{i_{\ell}+1}\right\}
$$

for some $s \in\{0, \ldots, n-2\}$ (where $A=\emptyset \Leftrightarrow s=0$ ). Notice that we always have $i_{\ell} \geqslant \ell$. Let

$$
\sigma=\sum_{i=1}^{n-2} i s_{i}\left(\varpi_{n-1}\right)=\sum_{i \in A} i=i_{1}+\cdots+i_{s}
$$

For each $j=0, \ldots, n-1$, let $\sigma_{n}(j) \in \mathbb{Z}_{n}$ be the class modulo $n$ of the difference

$$
\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}(j)\right)-\sigma
$$

We claim that for any $\varpi_{n-1} \in S_{n-1}$,

$$
\begin{equation*}
\left\{\sigma_{n}(j) \mid j \in\{0, \ldots, n-1\}\right\}=\{0, \ldots, n-1\} \tag{4.14}
\end{equation*}
$$

Note that this proves that $\mathcal{S}(n, d)$ is in bijection with $S_{n-1}$ for any $d$, hence proves the lemma.

To prove (4.14), we shall explicitly give for each $k \in\{0, \ldots, n-1\}$, the corresponding $j \in\{0, \ldots, n-1\}$ such that $\sigma_{n-1}(j)=k$.

- If $k=0$, then we have obviously to take $j=n-1$. Indeed, $\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}(n-\right.$ 1)) $=\sigma$, hence $\sigma_{n}(n-1)=0$.
- If $1 \leqslant k \leqslant s$, take $j=i_{\ell} \in A$, where $\ell=s-k+1$. Then $\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}\left(i_{\ell}\right)\right)=$ $\sigma+s-\ell+1=\sigma+k$, so $\sigma_{n}\left(i_{\ell}\right)=k$
- If $s+1 \leqslant k \leqslant s+i_{1}$, then take $j=k-s-1$. In fact, since $j<i_{1}$, we have that $\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}(j)\right)=j+1+\sigma+s=\sigma+k$, i.e., $\sigma_{n}(k-s-1)=k$.
- Suppose now that $i_{\ell}+s-\ell+2 \leqslant k \leqslant i_{\ell+1}+s-\ell$, for some $\ell \in\{1, \ldots, s-1\}$. Note that these situations are possible only if $i_{\ell+1} \geqslant i_{\ell}+2$. By taking $j=k-s+\ell-1$, one checks that $\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}(j)\right)=\sigma+j+s-\ell+1=\sigma+k$, and $\sigma_{n}(k-s+\ell-1)=k$.
- Finally, if $i_{s}+2 \leqslant k \leqslant n-1$, choose $j=k-1$. Indeed, $\sum_{i=1}^{n-1} i s_{i}\left(\varpi_{n-1}(j)\right)=$ $\sigma+j+1=\sigma+k$. Hence, $\sigma_{n}(k-1)=k$.
In these items we ran through all the possible values of $k \in\{0, \ldots, n-1\}$, exactly once each, and we found a bijection with the positions $j \in\{0, \ldots, n-1\}$ such that $\sigma_{n}(j)=k \in \mathbb{Z}_{n}$. This proves (4.14) and thus the lemma.

The following corollary proves the last statement of Theorem 3.15.
Corollary 4.7. If $n=2,3$, then (4.12) holds for any system of generic weights.
Proof. From Proposition 4.5 we know that (4.12) holds under the assumption of small weights. But by Corollary 3.10 we know that the $E$-polynomials $E(\mathcal{M})$ and $E(\mathcal{M})^{\Gamma_{n}}$ are independent of the (generic) weights, hence so is $E(\mathcal{M})^{\mathrm{var}}$ for any $n$. Theorem 3.14 implies then that $E\left(\mathcal{F}_{(1,1, \ldots, 1)}\right)^{\mathrm{var}}$ is also independent of the generic weights whenever $n=2,3$. Therefore the formula we reached is valid for any generic weights, for such $n$.

## 5. Unramified cyclic covers, norm maps and Pryms

The purpose of the following section is to recall some classical results about Prym varieties of unramified coverings, essentially going back to Narasimhan-Ramanan 31] and Mumford [30], and corresponding to Section 7 of [20]. For the benefit of the interested reader, we have included complete proofs.
5.1. Connected components of the kernel of a norm map. In Section 3.1, we considered the Prym variety of a ramified cover in the context of the Hitchin fibration. In the case of an unramified cover the structure of the kernel of the norm map turns out to be quite different.
Let $n$ be a prime number. Fix $\gamma \in \Gamma_{n}$ and let $L_{\gamma}$ be the corresponding $n$-torsion line bundle on $X$. Denote the associated unramified regular $n$-cover by

$$
\begin{equation*}
\pi: X_{\gamma} \rightarrow X \tag{5.1}
\end{equation*}
$$

Recall that $X_{\gamma}$ is the spectral cover of $X$ defined as the curve in the total space $\left|L_{\gamma}\right|$ of $L_{\gamma}$ defined by the equation $\lambda^{n}-1=0$, where $\lambda \in H^{0}\left(\left|L_{\gamma}\right|, p^{*} L_{\gamma}\right)$ is the tautological section, and $p:\left|L_{\gamma}\right| \rightarrow X$ is the projection. Then $\pi$ is the restriction of $p$ to $X_{\gamma}$. The line bundle $\pi^{*} L_{\gamma}$ is trivial over $X_{\gamma}$ since the nowhere vanishing section $\lambda: \mathcal{O}_{X_{\gamma}} \rightarrow \pi^{*} L_{\gamma}$ gives a canonical trivialization.

Let $\operatorname{Pic}(X)$ be the Picard group of $X$ and $\operatorname{Pic}^{i}(X)$ be the component corresponding to line bundles of degree $i$, so that $\operatorname{Pic}^{0}(X) \cong \operatorname{Jac}(X)$, and

$$
\operatorname{Pic}(X)=\bigsqcup_{i \in \mathbb{Z}} \operatorname{Pic}^{i}(X)
$$

Consider the same notations for the curve $X_{\gamma}$. The dimension of $\operatorname{Pic}(X)$ is $g$ while the dimension of $\operatorname{Pic}\left(X_{\gamma}\right)$ is the genus of $X_{\gamma}$, given by $n(g-1)+1$.

The pullback map $\pi^{*}: \operatorname{Pic}(X) \rightarrow \operatorname{Pic}\left(X_{\gamma}\right)$ is not injective neither surjective. The non-surjectivity of $\pi^{*}$ is clear by dimensional reasons and also because $\pi^{*}\left(\operatorname{Pic}^{i}(X)\right) \subset$ $\operatorname{Pic}^{n i}\left(X_{\gamma}\right)$. The next proposition provides the description of the image. Consider the Galois group of the covering $\pi: X_{\gamma} \rightarrow X$. It is isomorphic to $\mathbb{Z}_{n}$, which we consider as the group of the $n$-th roots of unity. Let $\xi=\exp (2 \pi i / n)$ be the standard generator. The Galois group $\mathbb{Z}_{n}$ acts on $\operatorname{Pic}^{i}\left(X_{\gamma}\right)$ by pullback and obviously a line bundle over $X_{\gamma}$ is fixed by $\mathbb{Z}_{n}$ if and only if it is fixed by $\xi$.
Proposition 5.1. The kernel of $\pi^{*}$ is the finite free abelian group generated by $L_{\gamma}$, that is $\operatorname{ker}\left(\pi^{*}\right) \cong \mathbb{Z}_{n}$. The image coincides with $\operatorname{Pic}\left(X_{\gamma}\right)^{\mathbb{Z}_{n}}$, i.e. the fixed point subvariety of $\operatorname{Pic}\left(X_{\gamma}\right)$ under the Galois group. So $\pi^{*}$ yields an isomorphism $\operatorname{Pic}^{i}(X) / \mathbb{Z}_{n} \cong \operatorname{Pic}^{n i}\left(X_{\gamma}\right)^{\mathbb{Z}_{n}}$.

Proof. We already know that $\pi^{*} L_{\gamma}^{j} \cong \mathcal{O}_{X_{\gamma}}$, for any $j=0, \ldots, n-1$. For the converse, take $L$ a degree 0 line bundle on $X$ whose pullback is trivial. Then

$$
\mathcal{O}_{X} \oplus L_{\gamma}^{-1} \oplus \cdots \oplus L_{\gamma}^{-(n-1)} \cong \pi_{*} \mathcal{O}_{X_{\gamma}} \cong \pi_{*} \pi^{*} L \cong L \otimes\left(\mathcal{O}_{X} \oplus L_{\gamma}^{-1} \oplus \cdots \oplus L_{\gamma}^{-(n-1)}\right)
$$

which implies that $L$ must be some power of $L_{\gamma}$.
Regarding the image of $\pi^{*}$, since $\pi \circ \xi=\pi$, it is clear that $\xi^{*} \pi^{*} L=\pi^{*} L$ for any $L \in \operatorname{Pic}(X)$. Conversely, if $F \in \operatorname{Pic}\left(X_{\gamma}\right)$ is fixed by $\xi$, then $F$ descends to a line bundle $L$ in $X$ so that $F=\pi^{*} L$.

We shall now consider the norm map associated to the unramified cover $\pi: X_{\gamma} \rightarrow X$. There are several incarnations of this map, all of them compatible with each other. We will consider three of them and use the same notation for all. The context will clarify the ones we are using. The norm map on divisors is given by

$$
\operatorname{Nm}_{\pi}: \operatorname{Div}\left(X_{\gamma}\right) \rightarrow \operatorname{Div}(X), \quad E=\sum_{p} E(p) p \mapsto \operatorname{Nm}_{\pi}(E)=\sum_{p} E(p) \pi(p) .
$$

(This has already been defined before, for more general coverings, in Definition 3.3.) We also have the norm map on the fields of non-zero meromorphic functions, given by

$$
\begin{equation*}
\operatorname{Nm}_{\pi}: \mathcal{M}\left(X_{\gamma}\right)^{*} \rightarrow \mathcal{M}(X)^{*}, \quad \operatorname{Nm}_{\pi}(f)(p)=\prod_{q \in \pi^{-1}(p)} f(q) \tag{5.2}
\end{equation*}
$$

It is clear that $\mathrm{Nm}_{\pi}(\operatorname{div}(f))=\operatorname{div}\left(\operatorname{Nm}_{\pi}(f)\right)$, for any $f \in \mathcal{M}\left(X_{\gamma}\right)^{*}$, hence the norm map on divisors induces the norm map on the Picard groups, i.e., on line bundles:

$$
\begin{equation*}
\operatorname{Nm}_{\pi}: \operatorname{Pic}\left(X_{\gamma}\right) \rightarrow \operatorname{Pic}(X), \quad \mathcal{O}_{X_{\gamma}}(E) \mapsto \mathcal{O}_{X}\left(\operatorname{Nm}_{\pi}(E)\right) \tag{5.3}
\end{equation*}
$$

Let $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ be the subvariety of $\operatorname{Jac}\left(X_{\gamma}\right)$ defined as the kernel of (5.3), and consider the group homomorphism

$$
\begin{equation*}
p: \operatorname{Pic}\left(X_{\gamma}\right) \rightarrow \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right), \quad L \mapsto L^{-1} \otimes \xi^{*} L \tag{5.4}
\end{equation*}
$$

It is well-defined since $\operatorname{Nm}_{\pi}\left(L^{-1} \otimes \xi^{*} L\right)=\operatorname{Nm}_{\pi}\left(L^{-1}\right) \otimes \operatorname{Nm}_{\pi}\left(\xi^{*} L\right)=\operatorname{Nm}_{\pi}(L)^{-1} \otimes$ $\operatorname{Nm}_{\pi}(L)=\mathcal{O}_{X}$.

The following is a generalization to $n \geqslant 2$ of Lemma 1 of Mumford [30].
Proposition 5.2. The homomorphism $p$ is surjective and the same holds for the restriction of $p$ to the disjoint union $\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$.

Proof. Let $M \in \operatorname{ker}\left(\operatorname{Nm}_{\pi}\right) \subset \operatorname{Jac}\left(X_{\gamma}\right)$. Then $M$ must be isomorphic to $\mathcal{O}_{X_{\gamma}}(F)$, for some degree 0 divisor $F$, such that $\operatorname{Nm}_{\pi}(F)=\operatorname{div}(f)$, for some non-zero meromorphic function $f$ on $X$. But the norm map (5.2) on function fields is surjective (see [27] and also [2, p. 282]), hence $f=\operatorname{Nm}_{\pi}(g)$ for some $g \in \mathcal{M}\left(X_{\gamma}\right)^{*}$. Let $G=\operatorname{div}(g)$. Then $\mathrm{Nm}_{\pi}(G)=\mathrm{Nm}_{\pi}(\operatorname{div}(g))=\operatorname{div}\left(\mathrm{Nm}_{\pi}(g)\right)=\operatorname{div}(f)=\mathrm{Nm}_{\pi}(F)$. Define $F=F-G$. Then $\operatorname{Nm}_{\pi}(\bar{F})=0$, hence $\bar{F}$ must be of the form $\bar{F}=\sum_{p \in X_{\gamma}} \bar{F}(p) p$ with

$$
\begin{equation*}
\sum_{i=0}^{n-1} \bar{F}\left(\xi^{i}(p)\right)=0 \tag{5.5}
\end{equation*}
$$

Now choose one and only one element in the support of $\bar{F}$ in each fibre of $\mathrm{Nm}_{\pi}$ over the support of $\mathrm{Nm}_{\pi}(\bar{F})$. This yields a collection of points $p_{1}, \ldots, p_{m}$ in the support of $\bar{F}$ such that $\pi\left(p_{i}\right) \neq \pi\left(p_{j}\right)$, for $i \neq j$. For each $l=1, \ldots, m$, choose integers $k_{1}(l), \ldots, k_{n}(l)$ such that

$$
\begin{equation*}
\bar{F}\left(\xi^{i}\left(p_{l}\right)\right)=-k_{i+1}(l)+k_{i}(l) \tag{5.6}
\end{equation*}
$$

for every $i=0, \ldots, n$ and where, by definition, $k_{0}(l)=k_{n}(l)$. This is possible (in an infinite number of ways) due to (5.5). Define the divisor

$$
\tilde{F}=\sum_{l=1}^{m} \sum_{i=0}^{n-1} k_{i+1}(l) \xi^{i}\left(p_{l}\right)
$$

and the corresponding line bundle $\tilde{L} \cong \mathcal{O}_{X_{\gamma}}(\tilde{F})$. It follows from (5.6) that $\mathcal{O}_{X_{\gamma}}(\bar{F}) \cong$ $\tilde{L}^{-1} \otimes \xi^{*} \tilde{L}$, that is,

$$
M \cong \tilde{L}^{-1} \otimes \xi^{*} \tilde{L}=p(\tilde{L})
$$

because $M \cong \mathcal{O}_{X_{\gamma}}(F), \bar{F}=F-G$ and $\mathcal{O}_{X_{\gamma}}(G)$ is trivial. Hence $p$ is surjective.
To show that its restriction to $\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$ is also surjective, consider the same line bundle $M=p(\tilde{L})$ and let $\tilde{d}$ be the degree of $\tilde{L}$. Let $a \in\{0, \ldots, n-1\}$ be the reduction of $\tilde{d}$ modulo $n$ and choose a line bundle $M^{\prime}$ on $X$, of degree $(\tilde{d}-a) / n$. Then $\operatorname{deg}\left(\tilde{L} \otimes \pi^{*} M^{\prime}\right)=a$ and

$$
M=\tilde{L}^{-1} \otimes \pi^{*} M^{\prime-1} \otimes \xi^{*}\left(\tilde{L} \otimes \pi^{*} M^{\prime}\right)=p\left(\tilde{L} \otimes \pi^{*} M^{\prime}\right)
$$

completing the proof.

Proposition 5.3. The kernel of $p$ equals the image of $\pi^{*}$. Hence

$$
\operatorname{ker}(p) \cap \bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)=\pi^{*}\left(\operatorname{Pic}^{0}(X)\right) \cong \operatorname{Pic}^{0}(X) / \mathbb{Z}_{n}
$$

Proof. The kernel of $p$ is precisely given by the fixed points under $\xi$ (thus under $\mathbb{Z}_{n}$ ), hence the result follows immediately from Proposition 5.1. The second part follows because $\pi^{*}\left(\operatorname{Pic}^{i}(X)\right) \subset \operatorname{Pic}^{n i}\left(X_{\gamma}\right)$.

The previous propositions can be summarized in the next corollary:
Corollary 5.4. The following sequence of groups is exact:

$$
0 \rightarrow \mathbb{Z}_{n} \rightarrow \operatorname{Pic}(X) \xrightarrow{\pi^{*}} \operatorname{Pic}\left(X_{\gamma}\right) \xrightarrow{p} \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right) \rightarrow 0 .
$$

Moreover, the restriction of $p$ to $\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$,

$$
\begin{equation*}
p: \bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right) \rightarrow \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right) \tag{5.7}
\end{equation*}
$$

is a holomorphic $\operatorname{Pic}^{0}(X) / \mathbb{Z}_{n}$-principal bundle.
The following is now immediate from the stated property of the map (5.7).
Corollary 5.5. The kernel $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ of the norm map (5.3) has $n$ connected components, which are labeled by the $n$ connected components of $\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$ via the group homomorphism (5.4).

Recall from Definition 3.4 that the Prym variety of $X_{\gamma}$ associated to the covering $\pi: X_{\gamma} \rightarrow X$ is the abelian variety defined as the connected component of $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ containing the identity:

$$
\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)=\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)_{0}
$$

Note that now we do not have the equality corresponding to (3.4).
Proposition 5.6. Two line bundles $M_{1}$ and $M_{2}$ are in the same connected component of the kernel of $\mathrm{Nm}_{\pi}$ if and only if $M_{1}=L_{1}^{-1} \otimes \xi^{*} L_{1}$ and $M_{2}=L_{2}^{-1} \otimes \xi^{*} L_{2}$, with $\operatorname{deg}\left(L_{1}\right)=\operatorname{deg}\left(L_{2}\right)$. In particular $\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$ is the subspace of $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ consisting of those line bundles of the form $L^{-1} \otimes \xi^{*} L$, with $\operatorname{deg}(L)=0$.

Thus the following sequence of groups is exact:

$$
0 \rightarrow \mathbb{Z}_{n} \rightarrow \operatorname{Pic}^{0}(X) \xrightarrow{\pi^{*}} \operatorname{Pic}^{0}\left(X_{\gamma}\right) \xrightarrow{p} \operatorname{Prym}_{\pi}\left(X_{\gamma}\right) \rightarrow 0,
$$

hence

$$
\begin{equation*}
\operatorname{dim}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right)=(n-1)(g-1) . \tag{5.8}
\end{equation*}
$$

We shall need a generalization of Proposition 5.6 to any fibre of the norm map, and not only its kernel. That is easily achieved since such fibre is a torsor for the kernel, hence being isomorphic to $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ although not canonically. Let then $\Lambda$ be a degree $d$, holomorphic line bundle over $X$. Choose an arbitrary line bundle $L_{0} \in \mathrm{Nm}_{\pi}^{-1}(\Lambda)$. Given this choice, we have the obvious isomorphism

$$
\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right) \stackrel{\cong}{\Rightarrow} \mathrm{Nm}_{\pi}^{-1}(\Lambda), \quad M \mapsto M \otimes L_{0} .
$$

Consider the union $\bigsqcup_{i=d}^{d+n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$. The same kind of isomorphism holds,

$$
\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right) \stackrel{\cong}{\rightrightarrows} \bigsqcup_{i=d}^{d+n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right), \quad L \mapsto L \otimes L_{0}
$$

and we have the analogue of the restriction of the map $p$ to $\bigsqcup_{i=0}^{n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right)$,

$$
\begin{equation*}
p_{L_{0}}: \bigsqcup_{i=d}^{d+n-1} \operatorname{Pic}^{i}\left(X_{\gamma}\right) \rightarrow \operatorname{Nm}_{\pi}^{-1}(\Lambda), \quad p_{L_{0}}(L)=L^{-1} \otimes \xi^{*} L \otimes L_{0} . \tag{5.9}
\end{equation*}
$$

Hence the following diagram commutes:


The twisted version of the previous results reads then as follows. This result (and the particular case of Corollary (5.5) goes back at least to Narasimhan-Ramanan [31].
Proposition 5.7. The map $p_{L_{0}}$ is a holomorphic $\operatorname{Pic}^{0}(X) / \mathbb{Z}_{n}$-principal bundle and the $n$ connected components of $\operatorname{Nm}_{\pi}^{-1}(\Lambda)$ are labeled by the degree $i \in\{d, \ldots, d+n-1\}$. Moreover, $\operatorname{Pic}^{i}(X)$ is a holomorphic $\operatorname{Pic}^{0}(X) / \mathbb{Z}_{n}$-principal bundle over a connected component of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$.

Thus each connected component of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ is a torsor for $\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$.
5.2. The action of the Galois group. We now wish to see how the Galois group $\mathbb{Z}_{n}$ acts on the on components of the fibre of the norm map. This is not strictly necessary for what follows but we include it for completeness.

We continue with an unramified $n$-cover (5.1) and a line bundle $\Lambda$ over $X$ of degree $d$. Let $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$ be the set consisting of the $n$ connected components of the fibre of the norm map of over $\Lambda$. Let $(n, d)$ denote the greatest common divisor of $n$ and $d$.

Proposition 5.8. The $\mathbb{Z}_{n}$-orbit of any element of $\pi_{0}\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda)\right)$ has $n /(n, d)$ elements. In particular, $\mathbb{Z}_{n}$ acts trivially on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$ if and only if $d$ is a multiple of $n$ and acts transitively if and only if $n$ and $d$ are coprime.

Proof. Let $M$ be a line bundle of degree $d$ such that $\mathrm{Nm}_{\pi}(M)=\Lambda$. Then $M=p_{L_{0}}(L)$ for some degree $i \in\{d, \ldots, d+n-1\}$ line bundle $L$ over $X_{\gamma}$, where $p_{L_{0}}$ is defined in (5.9),

$$
M=L^{-1} \otimes \xi^{*} L \otimes L_{0}
$$

By Proposition 5.7. the component of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ where $M$ lies is determined by the degree $i$ of $L$. Since

$$
\xi^{*} M=\xi^{*} L^{-1} \otimes \xi^{*} \xi^{*} L \otimes \xi^{*} L_{0}=\left(\xi^{*} L \otimes L_{0}\right)^{-1} \otimes \xi^{*}\left(\xi^{*} L \otimes L_{0}\right) \otimes L_{0}=p_{L_{0}}\left(\xi^{*} L \otimes L_{0}\right)
$$

then $\xi^{*} M$ lies in the component determined by the degree of $\xi^{*} L \otimes L_{0}$, which is $i+d$. So $\xi^{*} M$ is in the same connected component as $M$ is and only if $i+d$ is equal to $i$ modulo $n$, that is $d$ is a multiple of $n$.

If $n$ does not divide $d$ then, from what we saw, the orbit of $M$ on $\pi_{0}\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda)\right)$ is determined by class in $\mathbb{Z}_{n}$ of the numbers $i+j d$, with $j=0, \ldots, n-1$. Hence we conclude that the orbit of $M$ runs over $n /(n, d)$ different connected components of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$.

We know that $\operatorname{Pic}\left(X_{\gamma}\right)^{\mathbb{Z}_{n}}$ is the image of $\pi^{*}$. Let us now see what is its intersection with a fibre of $\mathrm{Nm}_{\pi}$.

Proposition 5.9. The intersection $\operatorname{Pic}\left(X_{\gamma}\right)^{\mathbb{Z}_{n}} \cap \mathrm{Nm}_{\pi}^{-1}(\Lambda)$ is the image of $\left.\pi^{*}\right|_{\Gamma_{n}}$. In particular, it is empty if $d$ is not a multiple of $n$. If not empty, it has $n^{2 g-1}$ elements.

Proof. Since $\mathbb{Z}_{n}$ is generated by $\xi$, it is enough to consider a line bundle $L \in \operatorname{Nm}_{\pi}^{-1}(\Lambda)$ such that $L \cong \xi^{*} L$. By Proposition 5.1, we have that $L \cong \pi^{*} N$ for some line bundle $N$ over $X$. Then $\Lambda \cong \operatorname{Nm}_{\pi}\left(\pi^{*} N\right) \cong N^{n}$. This is only possible if $d=n \operatorname{deg}(N)$, showing that otherwise the intersection is empty.

This proves that the intersection is the image, under $\pi^{*}$, of the $n^{2 g} n$ th-roots of $\Lambda$. But this image only has $n^{2 g-1}$ elements since two such roots are pulled-back to the same element whenever they differ by a power of $L_{\gamma}$.
5.3. The action of $\Gamma_{n}$ and the Weil pairing. Now we consider the action of $\Gamma_{n}$ on $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$. An element $\delta \in \Gamma_{n}$ acts on $M \in \operatorname{Nm}_{\pi}^{-1}(\Lambda)$ by

$$
\begin{equation*}
M \mapsto M \otimes \pi^{*} L_{\delta} \tag{5.10}
\end{equation*}
$$

where $L_{\delta}$ is the $n$-torsion line bundle on $X$ corresponding to $\delta$. Indeed, $\operatorname{Nm}_{\pi}\left(\pi^{*} L_{\delta}\right)=$ $L_{\delta}^{n} \cong \mathcal{O}_{X}$, therefore $\operatorname{Nm}_{\pi}\left(M \otimes \pi^{*} L_{\delta}\right)=\Lambda$.

Proposition 5.10. The subgroup $\mathbb{Z}_{n}=\langle\gamma\rangle \subset \Gamma_{n}$ acts trivially on $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$, and the $\Gamma_{n}$-action induces a free action of $\Gamma_{n} / \mathbb{Z}_{n} \cong \mathbb{Z}_{n}^{2 g-1}$ on $\operatorname{Nm}_{\pi}^{-1}(\Lambda)$.
Proof. The elements $\delta \in \Gamma_{n}$ which fix some element in $\operatorname{Nm}_{\pi}^{-1}(\Lambda)$ are the ones such that $\pi^{*} L_{\delta} \cong \mathcal{O}_{X_{\gamma}}$, and thus they fix every point in $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$. By Proposition 5.1, $\delta$ is such an element if and only if $L_{\delta}$ is a power of $L_{\gamma}$, i.e., $\delta \in\langle\gamma\rangle \cong \mathbb{Z}_{n}$. So the $\Gamma_{n}$-action factors through a free $\Gamma_{n} / \mathbb{Z}_{n} \cong \mathbb{Z}_{n}^{2 g-1}$-action on $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$.

We must especially study the action of $\Gamma_{n}$ on the set of connected components of $\operatorname{Nm}_{\pi}^{-1}(\Lambda)$. Since $\Gamma_{n} \cong H^{1}\left(X, \mathbb{Z}_{n}\right)$, we have a pairing on $\Gamma_{n}$ given by cup product followed by evaluation on the fundamental class:

$$
\begin{equation*}
\langle,\rangle: \Gamma_{n} \times \Gamma_{n} \rightarrow \mathbb{Z}_{n} \tag{5.11}
\end{equation*}
$$

where $\mathbb{Z}_{n}$ is given the multiplicative structure. This is a symplectic pairing, called the Weil pairing.

It will be convenient to give a different (but equivalent) definition of the Weil pairing. First, given a meromorphic function $f$ on $X$ and a divisor $D$ on $X$ whose support is disjoint from the support of the divisor of $f$, define

$$
\begin{equation*}
f(D)=\prod_{p \in X} f(p)^{D(p)} \tag{5.12}
\end{equation*}
$$

Weil reciprocity (see for instance [2, p. 283]) states that $f(\operatorname{div}(g))=g(\operatorname{div}(f))$ for any pair of meromorphic functions $f, g$ on $X$. Using this, the Weil pairing (5.11) can also be defined as follows. Take two $n$-torsion line bundles $L_{1}, L_{2}$ on $X$. Let $D_{1}$ and $D_{2}$ be divisors, with disjoint support, so that $L_{i} \cong \mathcal{O}_{X}\left(D_{i}\right)$. Then $n D_{i}=\operatorname{div}\left(f_{i}\right)$ for some meromorphic function $f_{i}: X \rightarrow \mathbb{C}$, and

$$
\begin{equation*}
\left\langle L_{1}, L_{2}\right\rangle=\frac{f_{1}\left(D_{2}\right)}{f_{2}\left(D_{1}\right)} \in \mathbb{Z}_{n} \tag{5.13}
\end{equation*}
$$

Remark 5.11. Sometimes (cf. [2]), $\left\langle L_{1}, L_{2}\right\rangle$ is defined as $\frac{n}{2 \pi i} \log \frac{f_{1}\left(D_{2}\right)}{f_{2}\left(D_{1}\right)}$, but this is by considering $\mathbb{Z}_{n}$ with the additive structure.

Recall that $\xi=\exp (2 \pi i / n) \in \mathbb{Z}_{n}$ denotes the standard generator of the multiplicative group $\mathbb{Z}_{n}$. Recall also that if $L_{\delta}$ is a $n$-torsion line bundle on $X$, then $\pi^{*} L_{\delta}$ lies in the kernel of the norm map, so by Proposition 5.2 it is of the form $F_{\delta}^{-1} \otimes \xi^{*} F_{\delta}$ for some line bundle $F_{\delta}$ on $X_{\gamma}$ of degree between 0 and $n-1$. The next result generalizes Mumford [30, Lemma 2] to $n \geqslant 2$.

Proposition 5.12. Let $L_{\gamma}$ be the line bundle corresponding to $\gamma \in \Gamma_{n} \backslash\{e\}$ and let $L_{\delta}$ be any n-torsion line bundle on $X$ which is not a power of $L_{\gamma}$. Let $F_{\delta}$ be a line bundle on $X_{\gamma}$, of degree between 0 and $n-1$, such that $\pi^{*} L_{\delta} \cong F_{\delta}^{-1} \otimes \xi^{*} F_{\delta}$. Then there exists a non-zero integer $l(\gamma)$ between 1 and $n-1$, depending only on $\gamma$, such that

$$
\left\langle L_{\delta}, L_{\gamma}\right\rangle=\xi^{l(\gamma) \operatorname{deg}\left(F_{\delta}\right)} .
$$

Proof. It will be convenient to use definition (5.13). Let $D_{\gamma}$ and $D_{\delta}$ be divisors on $X$, with disjoint support, such that $L_{\gamma} \cong \mathcal{O}_{X}\left(D_{\gamma}\right)$ and $L_{\delta} \cong \mathcal{O}_{X}\left(D_{\delta}\right)$ and such that $n D_{\gamma}=\operatorname{div}\left(f_{\gamma}\right)$ and $n D_{\delta}=\operatorname{div}\left(f_{\delta}\right)$ for some non-zero meromorphic functions $f_{\gamma}, f_{\delta}$. Since $\pi^{*} L_{\gamma}$ is trivial, then $\pi^{*} D_{\gamma}=\operatorname{div}(g)$ for some meromorphic function $g \in \mathcal{M}\left(X_{\gamma}\right)^{*}$. From this it follows that $f_{\gamma}=\mathrm{Nm}_{\pi}(g)$. On the other hand, if $F_{\delta} \cong \mathcal{O}_{X_{\gamma}}(D)$ for some divisor $D$ on $X_{\gamma}$, then there is a non-zero meromorphic function $h$ on $X_{\gamma}$ such that $\pi^{*} D_{\delta}=-D+\xi D+\operatorname{div}(h)$. It also follows that $f_{\delta}=\operatorname{Nm}_{\pi}(h)$.

Since $\operatorname{div}\left(\xi^{*} g\right)=\xi^{*} \pi^{*} D_{\gamma}=\pi^{*} D_{\gamma}=\operatorname{div}(g)$, there is a non-zero complex number $\lambda(\gamma)$, depending only on $g$, i.e. only on $\gamma$, such that

$$
\begin{equation*}
\xi^{*} g=\lambda(\gamma) g . \tag{5.14}
\end{equation*}
$$

Note that $\lambda(\gamma) \neq 1$, since otherwise that would be saying that $L_{\gamma}$ was trivial. Furthermore, $g^{n}=\pi^{*} f_{\gamma}=\xi^{*} \pi^{*} f_{\gamma}=\left(\xi^{*} g\right)^{n}$, hence $\lambda(\gamma)^{n}=1$, thus $\lambda(\gamma)=\xi^{l(\gamma)}$ for some integer $l(\gamma) \in\{0, \ldots, n-1\}$ depending only on $\gamma$.
Recall that $\operatorname{Nm}_{\pi}(g)(p)=\prod_{\tilde{p} \in \pi^{-1}(p)} g(\tilde{p})$. Then it is easy to see directly from the definition (5.12), that $\mathrm{Nm}_{\pi}(g)\left(D_{\delta}\right)=g\left(\pi^{*} D_{\delta}\right)$. Similarly for $\mathrm{Nm}_{\pi}(h)\left(D_{\gamma}\right)$. Then, using (5.14), we get

$$
\begin{aligned}
\left\langle L_{\delta}, L_{\gamma}\right\rangle & =\frac{\operatorname{Nm}_{\pi}(h)\left(D_{\gamma}\right)}{\operatorname{Nm}_{\pi}(g)\left(D_{\delta}\right)}=\frac{h\left(\pi^{*} D_{\gamma}\right)}{g\left(\pi^{*} D_{\delta}\right)} \\
& =\frac{h(\operatorname{div}(g))}{g(\operatorname{div}(h)-D+\xi D)}=\frac{g(D)}{g(\xi D)} \\
& =\frac{\prod_{p \in X_{\gamma}} g(p)^{D(p)}}{\prod_{p \in X_{\gamma}} g(p)^{(\xi D)(p)}}=\frac{\prod_{p \in X_{\gamma}} g(p)^{D(p)}}{\prod_{p \in X_{\gamma}}\left(\lambda(\gamma)^{-1} g(\xi \cdot p)\right)^{D(\xi \cdot p)}} \\
& =\lambda(\gamma)^{\sum_{p \in X_{\gamma}} D(\xi \cdot p)}=\lambda(\gamma)^{\operatorname{deg}\left(F_{\delta}\right)} \\
& =\xi^{l(\gamma) \operatorname{deg}\left(F_{\delta}\right)}
\end{aligned}
$$

as claimed.
If $n=2$ we have $l(\gamma)=1$ for every non-trivial $\gamma$, recovering Mumford's result.
The Weil pairing corresponds to intersection form in homology $H_{1}\left(X, \mathbb{Z}_{n}\right)$. From this definition it is clear that, for each $\gamma \in \Gamma_{n} \backslash\{e\}$, it is possible to choose a basis

$$
\begin{equation*}
\left(\gamma, \delta_{0}, \delta_{1}, \ldots, \delta_{2 g-2}\right) \tag{5.15}
\end{equation*}
$$

of $\Gamma_{n}$ including $\gamma$, and such that

$$
\left\langle L_{\delta_{i}}, L_{\gamma}\right\rangle= \begin{cases}\xi^{l(\gamma)} & i=0  \tag{5.16}\\ 1 & i \neq 0\end{cases}
$$

We assume from now on that such a basis has been chosen.
Now we describe the $\Gamma_{n}$-action on the $n$ connected components of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$.
Proposition 5.13. Every element of $\Gamma_{n}$ which is not in the subgroup $\left\langle\delta_{0}\right\rangle$ generated by $\delta_{0}$ acts trivially on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$, and $\left\langle\delta_{0}\right\rangle \cong \mathbb{Z}_{n}$ acts freely and transitively on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$. In particular, $\Gamma_{n}$ acts transitively on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$.

Proof. We will use Proposition 5.7. Let $M \in \operatorname{Nm}_{\pi}^{-1}(\Lambda)$, so that $M=p_{L_{0}}(L)$ for some degree $i \in\{d, \ldots, d+n-1\}$ line bundle $L$ over $X_{\gamma}$, i.e.

$$
M=L^{-1} \otimes \xi^{*} L \otimes L_{0}
$$

Since $\pi^{*} L_{\delta} \in \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ then, by Proposition 5.2, $\pi^{*} L_{\delta}=p\left(F_{\delta}\right)=F_{\delta}^{-1} \otimes \xi^{*} F_{\delta}$ for some line bundle $F_{\delta}$ on $X_{\gamma}$ of degree between 0 and $n-1$. So

$$
M \otimes \pi^{*} L_{\delta}=\left(L \otimes F_{\delta}\right)^{-1} \otimes \xi^{*}\left(L \otimes F_{\delta}\right) \otimes L_{0}
$$

If $\delta \notin\left\langle\delta_{0}\right\rangle$, then by (5.16), $\left\langle L_{\delta}, L_{\gamma}\right\rangle=1$ hence, by Proposition 5.12, $\operatorname{deg}\left(F_{\delta}\right)=0$, so that $\operatorname{deg}\left(L \otimes F_{\delta}\right)=i$ and therefore $M \otimes \pi^{*} L_{\delta}$ is in the same connected component as $M$ by Proposition 5.7. For $\delta_{0}$, for the same reasons we have $\operatorname{deg}\left(F_{\delta}\right)=1$, so $\operatorname{deg}\left(L \otimes F_{\delta}\right)=i+1$, thus the powers of $\delta_{0}$ act transitively on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$.

## 6. The polynomial $E_{\mathrm{st}}\left(\mathcal{M} / \Gamma_{n}\right)-E(\mathcal{M})^{\Gamma_{n}}$

6.1. The subvarieties $\mathcal{M}^{\gamma}$. The stringy $E$-polynomial of $\mathcal{M} / \Gamma_{n}$ is defined by (3.11), but the statement of Theorem 3.15 is only about

$$
\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}
$$

This is the polynomial we aim to compute in the present section. This requires the study of the subvarieties $\mathcal{M}^{\gamma}$ of fixed points under the action of each nontrivial $\gamma \in \Gamma_{n}$. This study is an adaptation to parabolic Higgs bundles of the corresponding result for vector bundles studied by Narasimhan and Ramanan in [31] (cf. Hausel and Thaddeus [20, Sec. 7] for the case of Higgs bundles in the non-parabolic situation).

Recall that to each non-trivial $\gamma \in \Gamma_{n}$ we associate an unramified cyclic $n$-cover $\pi$ : $X_{\gamma} \rightarrow X$ as in (5.1), with Galois group isomorphic to $\mathbb{Z}_{n}$ and $\mathrm{Nm}_{\pi}: \operatorname{Pic}^{d}\left(X_{\gamma}\right) \rightarrow \operatorname{Pic}^{d}(X)$ is the corresponding norm map.

Let $D_{\gamma}=\pi^{-1}(D)$ be the inverse image in $X_{\gamma}$ of our fixed divisor $D=p_{1}+\cdots+p_{|D|}$ in $X$, from which we have our fixed generic parabolic type $\boldsymbol{\alpha}$,

$$
0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1,
$$

for each $p \in D$. It is important to note that our genericity assumption on the parabolic type $\boldsymbol{\alpha}$ still holds, but that we do not need them to be small as it was required to prove in Proposition 4.5 the equality corresponding to the left-hand side of Theorem 3.15.

Given

$$
\begin{equation*}
\varpi_{n}=\left(\varpi_{n}\left(p_{1}\right), \ldots, \varpi_{n}\left(p_{|D|}\right)\right) \in S_{n}^{|D|}, \tag{6.1}
\end{equation*}
$$

we naturally construct a parabolic type of rank 1 , denoted by $\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)$ on $D_{\gamma}$ as follows. For each $p \in D$, write

$$
\varpi_{n}(p)=a_{1}(p) a_{2}(p) \ldots a_{n}(p) \in S_{n} .
$$

Then attach the weights $0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1$ to the set $\pi^{-1}(p)=\left\{q_{1}, \ldots, q_{n}\right\} \subset$ $D_{\gamma}$ so that the point $q_{i} \in \pi^{-1}(p)$ is given the weight $\alpha_{a_{i}(p)}(p)$. This yields the parabolic type $\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)$,

$$
0 \leqslant \alpha_{a_{i}(p)}(p)<1
$$

at each $q_{i} \in \pi^{-1}(p) \subset D_{\gamma}$. Strictly speaking, this depends on a choice of an ordering of the points in $\pi^{-1}(p)$ for each $p$. This ordering was implicitly chosen when we wrote $\pi^{-1}(p)=\left\{q_{1}, \ldots, q_{n}\right\}$. So, without the choice of that ordering, $\varpi_{n}$ in (6.1) belongs to a torsor for the group $S_{n}^{|D|}$. In any case, any ordering is valid for our purposes.

For each $\varpi_{n} \in S_{n}^{|D|}$, consider the moduli space $\mathcal{M}_{d}^{\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)}\left(\mathbb{C}^{*}\right)$ of strongly parabolic Higgs line bundles (i.e., $\mathbb{C}^{*}$-Higgs bundles) of degree $d \in \mathbb{Z}$, over $X_{\gamma}$, of parabolic type $\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)$ over $D_{\gamma}$. Let $K_{\gamma}=\pi^{*} K$ be the canonical line bundle of $X_{\gamma}$.

## Lemma 6.1.

(1) For every $\varpi_{n} \in S_{n}^{|D|}$, the moduli space $\mathcal{M}_{d}^{\alpha_{\gamma}\left(\varpi_{n}\right)}\left(\mathbb{C}^{*}\right)$ is isomorphic to the moduli space of Higgs line bundles of degree d over $X_{\gamma}$, i.e., to the cotangent bundle $T^{*} \operatorname{Pic}^{d}\left(X_{\gamma}\right) \cong \operatorname{Pic}^{d}\left(X_{\gamma}\right) \times H^{0}\left(X_{\gamma}, K_{\gamma}\right)$.
(2) The disjoint union $\bigsqcup_{w_{n} \in S_{n}^{|D|}} \mathcal{M}_{d}^{\boldsymbol{\alpha}_{\gamma}\left(\omega_{n}\right)}\left(\mathbb{C}^{*}\right)$ is isomorphic to $T^{*} \operatorname{Pic}^{d}\left(X_{\gamma}\right) \times S_{n}^{|D|}$.

Proof. Take a strongly parabolic Higgs line bundle $(F, \phi) \in \mathcal{M}_{d}^{\boldsymbol{\alpha}_{\gamma}\left(\omega_{n}\right)}\left(\mathbb{C}^{*}\right)$. Then $F \in$ $\operatorname{Pic}^{d}\left(X_{\gamma}\right)$ and $\phi \in H^{0}\left(X_{\gamma}, K_{\gamma}\left(D_{\gamma}\right)\right)$. But the fact that $\phi$ is strongly parabolic and $F$ is a line bundle implies that actually $\phi \in H^{0}\left(X, K_{\gamma}\right)$, thus the map that forgets the parabolic structure yields the isomorphism stated in (1). Then (2) follows from (1).

Proposition 6.2. Let $\boldsymbol{\alpha}=\left(\alpha_{1}(p), \ldots, \alpha_{n}(p)\right)_{p \in D}$ be any generic parabolic type and let $\varpi_{n} \in S_{n}^{|D|}$. If $(F, \phi) \in \mathcal{M}_{d}^{\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)}\left(\mathbb{C}^{*}\right)$, then $\left(\pi_{*} F, \pi_{*} \phi\right)$ is a semistable strongly parabolic $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle of degree $d$ and parabolic type $\boldsymbol{\alpha}$.

Proof. Let $V=\pi_{*} F$ and $\varphi=\pi_{*} \phi: V \rightarrow V \otimes K$. Since $\pi$ is unramified and $F$ has degree $d$, then so has the rank $n$ vector bundle $V$. Note that $K$ is a subsheaf of $K(D)$ so $\varphi$ is also a section of $\operatorname{End}(V) \otimes K(D)$. Let us see that $V$ has a parabolic structure at $D$ of type $\boldsymbol{\alpha}$. Let $p \in D$ and $\left\{q_{1}, \ldots, q_{n}\right\}=\pi^{-1}(p) \subset D_{\gamma}$. The filtration of $V_{p}$ is given as

$$
\begin{equation*}
V_{p}=V_{p, 1}=\bigoplus_{i=1}^{n} F_{q_{i}} \supsetneq V_{p, 2} \supsetneq \cdots \supsetneq V_{p, n} \supsetneq\{0\}, \quad 0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1, \tag{6.2}
\end{equation*}
$$

where, for $2 \leqslant j \leqslant n$,

$$
\begin{equation*}
V_{p, j}=V_{p, j-1} / F_{q^{\prime}} \tag{6.3}
\end{equation*}
$$

with $q^{\prime}$ being the point $q_{i}$ attached with the weight $\alpha_{j-1}(p)$ i.e. $q^{\prime}=q_{i}$ such that $a_{i}(p)=$ $j-1$. So $V_{p, n}=F_{q_{i}}$ such that $a_{i}(p)=n, F_{p, n-1}=F_{q_{j}} \oplus F_{q_{i}}$ with $j$ such that $a_{j}=n-1$, and so on. Doing this for every point of $D$ determines the parabolic structure of $V$ of type $\boldsymbol{\alpha}$.

Notice that, conversely, the parabolic structure of $V$, given at each point of $D$ by (6.2) and (6.3) determines the element (6.1) of $S_{n}^{|D|}$.

Over each $p \in D, \varphi_{p}=\left.\varphi\right|_{V_{p}}$ is diagonal with respect to the decomposition $V_{p}=\bigoplus F_{q_{i}}$ of $V_{p}$. Look at $\phi \in H^{0}\left(X_{\gamma}, K_{\gamma}\right)$ as a section of $K_{\gamma}\left(D_{\gamma}\right)$ which vanishes at every $q_{i} \in D_{\gamma}$. Then it is clear that $\varphi$ is strongly parabolic with respect to (6.2). So $(V, \varphi)$ is a strongly parabolic Higgs bundle of rank $n$, degree $d$ and parabolic type $\boldsymbol{\alpha}$.

It remains to check semistability. For that, recall that $\xi=\exp (2 \pi i / n)$ denotes the standard generator of the Galois group $\mathbb{Z}_{n}$, and note that $\left(\pi^{*} V, \pi^{*} \varphi\right) \cong\left(F \oplus \xi^{*} F \oplus \cdots \oplus\right.$ $\left.\xi^{* n-1} F, \phi \oplus \xi^{*} \phi \oplus \cdots \oplus \xi^{* n-1} \phi\right)$ is a strongly parabolic GL( $n, \mathbb{C}$ )-Higgs bundle on $X_{\gamma}$ with parabolic structure over $D_{\gamma}$ induced from (6.2). Since

$$
\operatorname{par} \mu\left(\pi^{*} \pi_{*} F\right)=\frac{n d+\sum_{q \in D_{\gamma}} \sum_{i=1}^{n} \alpha_{i}(\pi(q))}{n}=d+\sum_{p \in D} \sum_{i=1}^{n} \alpha_{i}(p)=\operatorname{par} \mu\left(\xi^{j *} F\right),
$$

for every $j=0, \ldots, n-1$, then $\pi^{*} V \cong \bigoplus_{j=1}^{n} \xi^{j *} F$ is a direct sum of parabolic line bundles, all of the same slope, and therefore it is semistable [35, Corollaire 10, p. 71]. Take a $\varphi$-invariant subbundle $V^{\prime} \subset V$ of degree $d^{\prime}$ and rank $n^{\prime}$. Then $\pi^{*} V^{\prime}$ is a $\pi^{*} \varphi$-invariant
subbundle of $\pi^{*} V$. By semistability of $\pi^{*} V$, we must have $\operatorname{par} \mu\left(\pi^{*} V^{\prime}\right) \leqslant \operatorname{par} \mu\left(\pi^{*} V\right)$, that is

$$
\frac{n d^{\prime}+n \sum_{p \in D} \sum_{i=1}^{n} \alpha_{i}(p)}{n^{\prime}} \leqslant \frac{n d+n \sum_{p \in D} \sum_{i=1}^{n} \alpha_{i}(p)}{n} .
$$

But this is equivalent to

$$
\frac{d^{\prime}+\sum_{p \in D} \sum_{i=1}^{n} \alpha_{i}(p)}{n^{\prime}} \leqslant \frac{d+\sum_{p \in D} \sum_{i=1}^{n} \alpha_{i}(p)}{n},
$$

that is, to $\operatorname{par} \mu\left(V^{\prime}\right) \leqslant \operatorname{par} \mu(V)$, proving semistability of $(V, \varphi)$.
So the push-forward gives a map from $\bigsqcup_{\varpi_{n} \in S_{n}^{|D|}} \mathcal{M}_{d}^{\boldsymbol{\alpha}_{\gamma}\left(\varpi_{n}\right)}\left(\mathbb{C}^{*}\right) \cong T^{*} \operatorname{Pic}^{d}\left(X_{\gamma}\right) \times S_{n}^{|D|}$ to the moduli space $\mathcal{M}_{d}^{\alpha}(\operatorname{GL}(n, \mathbb{C}))$ of degree $d$, strongly parabolic GL $(n, \mathbb{C})$-Higgs bundles, with parabolic type $\boldsymbol{\alpha}$. If we want the determinant to be $\Lambda$, we have to restrict this map to the subspace $T^{*} \mathrm{Nm}_{\pi}^{-1}(\Lambda) \times S_{n}^{|D|}$ if $n \geqslant 3$ is prime (or to $T^{*} \mathrm{Nm}_{\pi}^{-1}\left(\Lambda L_{\gamma}\right) \times$ $S_{2}^{|D|}$ if $n=2$ ), where $T^{*} \mathrm{Nm}_{\pi}^{-1}(\Lambda)$ denotes the cotangent bundle to $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$. Indeed, $\operatorname{det}\left(\pi_{*} F\right) \cong \operatorname{Nm}_{\pi}(F) L_{\gamma}^{-n(n-1) / 2}$, so if $n \geqslant 3$ is prime, $\operatorname{Nm}_{\pi}(F) \cong \Lambda$ (and if $n=2$, $\left.\mathrm{Nm}_{\pi}(F) \cong \Lambda L_{\gamma}\right)$. Thus $\left(\pi_{*} F, \pi_{*} \phi\right)$ is such that $\operatorname{det}\left(\pi_{*} F\right) \cong \Lambda$ and $\operatorname{tr}\left(\pi_{*} \phi\right)=0$ if and only if $(F, \phi) \in T^{*} \mathrm{Nm}_{\pi}^{-1}(\Lambda) \times S_{n}^{|D|}$, if $n \geqslant 3$ prime, or $(F, \phi) \in T^{*} \mathrm{Nm}_{\pi}^{-1}\left(\Lambda L_{\gamma}\right) \times S_{2}^{|D|}$ if $n=2$. In any case, $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ is a torsor for $\operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ thus $T^{*} \mathrm{Nm}_{\pi}^{-1}(\Lambda)$ is also a torsor for $T^{*} \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$. In turn, from Proposition 5.7, each connected component of $T^{*} \operatorname{ker}\left(\mathrm{Nm}_{\pi}\right)$ is a torsor for $T^{*} \operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$ which is isomorphic to $\operatorname{Prym}_{\pi}\left(X_{\gamma}\right) \times \mathbb{C}^{(n-1)(g-1)}$, since the Prym is an abelian variety of dimension $(n-1)(g-1)$, by (5.8). Hence

$$
T^{*} \operatorname{Nm}_{\pi}^{-1}(\Lambda) \cong \operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)}
$$

Therefore, Proposition 6.2 gives a map

$$
\begin{equation*}
\pi_{*}: \operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|} \rightarrow \mathcal{M} \tag{6.4}
\end{equation*}
$$

with the obvious modification when $n=2$.
Now we can describe the locus $\mathcal{M}^{\gamma}$ of points in $\mathcal{M}$ fixed by a non-trivial element $\gamma \in \Gamma_{n}$. This locus is going to be the image of $\pi_{*}$, which is isomorphic to the quotient of its domain by a natural action of the Galois group of $\pi: X_{\gamma} \rightarrow X$.

Theorem 6.3. Let $\boldsymbol{\alpha}$ be any generic parabolic type and let $n \geqslant 3$ be prime. For every $\gamma \in \Gamma_{n} \backslash\{e\}$, the map (6.4) induces an isomorphism

$$
\mathcal{M}^{\gamma} \simeq\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}\right) / \mathbb{Z}_{n}
$$

with $\mathbb{Z}_{n}$ acting diagonally, by pullback on $T^{*} \operatorname{Nm}_{\pi}^{-1}(\Lambda)$ and cyclically on each factor of $S_{n}^{|D|}$. If $n=2$, replace $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ by $\mathrm{Nm}_{\pi}^{-1}\left(\Lambda L_{\gamma}\right)$.
Proof. Let $\gamma \in \Gamma_{n} \backslash\{e\}$ and $(V, \varphi)$ represent a point in $\mathcal{M}$. Since $(V, \varphi)$ is stable then its only parabolic endomorphisms are the scalars [39, (3.3)], hence the same argument as in Proposition 2.6 of [31] shows that $V \cong V \otimes L_{\gamma}$ if and only if isomorphic to the push-forward of a line bundle $F$ over $X_{\gamma}$,

$$
V \cong \pi_{*} F .
$$

Moreover, the isomorphism $V \cong V \otimes L_{\gamma}$ is given by

$$
\begin{equation*}
\pi_{*}\left(\operatorname{Id}_{F} \otimes \lambda\right): \pi_{*} F \stackrel{\cong}{\rightrightarrows} \pi_{*} F \otimes L_{\gamma} \tag{6.5}
\end{equation*}
$$

where we recall that $\lambda: \mathcal{O}_{X_{\gamma}} \rightarrow \pi^{*} L_{\gamma}$ denotes the tautological section. From here one sees that the Higgs fields on $V \cong \pi_{*} F$ which are compatible under the isomorphism (6.5) are the ones which are push-forward of Higgs fields on $F$.

Consider now the parabolic structure on $V \cong \pi_{*} F$,

$$
\begin{equation*}
V_{p}=V_{p, 1} \supsetneq V_{p, 2} \supsetneq \cdots \supsetneq V_{p, n} \supsetneq\{0\}, \quad 0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1 \tag{6.6}
\end{equation*}
$$

over each $p \in D$, so that the one on $V \otimes L_{\gamma}$ is
$V_{p} \otimes L_{\gamma}=V_{p, 1} \otimes L_{\gamma, p} \supsetneq V_{p, 2} \otimes L_{\gamma, p} \supsetneq \cdots \supsetneq V_{p, n} \otimes L_{\gamma, p} \supsetneq\{0\}, \quad 0 \leqslant \alpha_{1}(p)<\cdots<\alpha_{n}(p)<1$.
For $p \in D$, let $\pi^{-1}(p)=\left\{q_{1}, \ldots, q_{n}\right\}$. Then the isomorphism (6.5) over $p$ is

$$
\begin{equation*}
V_{p}=\left(\pi_{*} F\right)_{p}=\bigoplus_{i=1}^{n} F_{q_{i}} \xrightarrow{\oplus_{i=1}^{n}\left(\operatorname{Id}_{F_{q_{i}}} \otimes \lambda_{q_{i}}\right)} \bigoplus_{i=1}^{n} F_{q_{i}} \otimes L_{\gamma, p}=V_{p} \otimes L_{\gamma, p} \tag{6.8}
\end{equation*}
$$

Since $q_{i} \neq q_{j}$ then also $\lambda\left(q_{i}\right) \neq \lambda\left(q_{j}\right)$ for $i \neq j$. Hence the only non-trivial subspaces $0 \neq V_{p}^{\prime}$ of $V_{p}$ which are preserved under (6.8) those of the form

$$
\begin{equation*}
V_{p}^{\prime}=\bigoplus_{j=1}^{\operatorname{dim}\left(V_{p}^{\prime}\right)} F_{q_{i j}} \tag{6.9}
\end{equation*}
$$

So the isomorphism (6.5) respects the filtrations (6.6) and (6.7) if and only if each $V_{p, i}$ is of the form (6.9).

So, after providing an element $\varpi_{n} \in S_{n}^{|D|}$, this parabolic structure of $V$ determines a parabolic structure on $F$ over $D_{\gamma}=\pi^{-1}(D)$, by reversing the construction carried in (6.2) and (6.3).

The conclusion is that $(V, \varphi) \in \mathcal{M}^{\gamma}$, i.e., $(V, \varphi) \cong\left(V \otimes L_{\gamma}, \varphi \otimes \operatorname{Id}_{L_{\gamma}}\right)$ if and only if $(V, \varphi) \cong \pi_{*}\left((F, \phi), \varpi_{n}\right)$ for some $\left((F, \phi), \varpi_{n}\right) \in \mathrm{Nm}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}$ (with the obvious modification if $n=2$ ).

It turns out that there are redundancies coming precisely from the action of the Galois group $\mathbb{Z}_{n}$ of $\pi: X_{\gamma} \rightarrow X$. Then clearly, for any $j=0, \ldots, n-1$, we have $\pi_{*}(F, \phi) \cong$ $\pi_{*}\left(\xi^{j *} F, \xi^{j *} \phi\right)$ as (non-parabolic) Higgs bundles and these are the only redundancies. Now we take into account the parabolic structure. A parabolic Higgs bundle is defined by $\left((F, \phi), \varpi_{n}\right) \in \mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}$. Consider the action of the Galois group $\mathbb{Z}_{n}$ given by

$$
\begin{equation*}
\xi^{j} \cdot\left((F, \phi), \varpi_{n}\right)=\left(\left(\xi^{j *} F, \xi^{j *} \phi\right), \xi^{-j} \cdot \varpi_{n}\right) \tag{6.10}
\end{equation*}
$$

where each $\xi^{j} \in \mathbb{Z}_{n}$ acts diagonally on $\varpi_{n}$ such that on each factor $\varpi_{n}(p)$ it acts as a cyclic permutation of length $i$ (hence acts freely). Precisely, $\xi^{j}$ acts diagonally on $\varpi_{n}$ as

$$
\begin{equation*}
\xi^{-j} \cdot \varpi_{n}(p)=a_{n-j+1}(p) \ldots a_{n}(p) a_{1}(p) \ldots a_{n-j}(p) \tag{6.11}
\end{equation*}
$$

for each $p \in D$. Thus the orbit of $\varpi_{n}$ is given by the set of all the $n^{|D|}$ permutations which differ from the given one at each point by a cyclic permutation.

It is easy to check, by following again the construction in (6.2) and (6.3), that two elements of $\mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}$ give rise to isomorphic parabolic Higgs bundles if and only if they are in the same orbit under $\mathbb{Z}_{n}$. In other words, $\left(\pi_{*} F, \pi_{*} \phi, \varpi_{n}\right)$ and $\left(\pi_{*} \xi^{j *} F, \pi_{*} \xi^{j *} \phi, \xi^{-j} \cdot \varpi_{n}\right)$ determine isomorphic parabolic Higgs bundles, for each $j=$ $0, \ldots, n-1$, and that is the only way one can obtain isomorphic parabolic Higgs bundles under our construction. We conclude that

$$
\mathcal{M}^{\gamma} \simeq\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}\right) / \mathbb{Z}_{n}
$$

with $\mathbb{Z}_{n}$ acting diagonally as in (6.10) and (6.11).
Remark 6.4. As mentioned in Remark 3.11, our description of $\mathcal{M}^{\gamma}$ implies that the corresponding parabolic Higgs bundles are $\boldsymbol{\alpha}$-semistable for any value of $\boldsymbol{\alpha}$.

In particular, it follows from this theorem that, for any non-trivial $\gamma \in \Gamma_{n}$,

$$
\begin{equation*}
\operatorname{dim}\left(\mathcal{M}^{\gamma}\right)=2(n-1)(g-1) \tag{6.12}
\end{equation*}
$$

It turns out that the parabolic structure will now make our life easier by allowing a slightly different description of the fixed point locus $\mathcal{M}^{\gamma}$, from which the calculation of the $\Gamma_{n}$-invariant $E$-polynomial $E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}$ is simpler than in the non-parabolic case.

First, choose a section

$$
\begin{equation*}
s: S_{n}^{|D|} / \mathbb{Z}_{n} \rightarrow S_{n}^{|D|} \tag{6.13}
\end{equation*}
$$

of the projection $S_{n}^{|D|} \rightarrow S_{n}^{|D|} / \mathbb{Z}_{n}$. Of course it corresponds to the choice of a representative of each class in $S_{n}^{|D|} / \mathbb{Z}_{n}$. There is no canonical choice of such $s$, but all of them are obviously algebraic.

Recall that $\Gamma_{n}$ acts on $\mathcal{M}^{\gamma}$, by acting trivially on the Higgs field and on the weights and by pullback and tensor product on the factor $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ of $\mathcal{M}^{\gamma}$; cf. (5.10).

Proposition 6.5. There is a $\Gamma_{n}$-equivariant isomorphism (depending on the choice of the section $s$ in (6.13))

$$
\mathcal{M}^{\gamma} \simeq \operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times\left(S_{n}^{|D|} / \mathbb{Z}_{n}\right)
$$

where $\Gamma_{n}$ acts on the first factor as stated in (5.10) and trivially on the other two factors.
Proof. From Theorem 6.3, we know that $\mathcal{M}^{\gamma} \simeq\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}\right) / \mathbb{Z}_{n}$. Consider the map

$$
f_{s}: \mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times\left(S_{n}^{|D|} / \mathbb{Z}_{n}\right) \rightarrow\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}\right) / \mathbb{Z}_{n}
$$

defined by

$$
f_{s}\left(F, \phi,\left[\varpi_{n}\right]\right)=\left[\left(F, \phi, s\left(\left[\varpi_{n}\right]\right)\right)\right] .
$$

Its inverse $g_{s}$ is defined as follows. Take $\left[\left(F, \phi, \varpi_{n}\right)\right] \in\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}\right) / \mathbb{Z}_{n}$. Since $\mathbb{Z}_{n}$ acts freely on $S_{n}^{|D|}$, there is a unique $i$ such that $\xi^{i} \cdot \varpi_{n}=s\left(\left[\varpi_{n}\right]\right)$. Then $\left[\left(F, \phi, \varpi_{n}\right)\right]=\left[\left(\xi^{i} \cdot F, \xi^{i} \cdot \phi, \xi^{i} \cdot \varpi_{n}\right)\right]$, and hence take

$$
g_{s}\left(\left[\left(F, \phi, \varpi_{n}\right)\right]\right)=\left(\xi^{i} \cdot F, \xi^{i} \cdot \phi,\left[\varpi_{n}\right]\right)
$$

It is clear that indeed $g_{s}=f_{s}^{-1}$. It is clear that both $f_{s}$ and its inverse are algebraic, yielding the stated isomorphism.

To see that it is $\Gamma_{n}$-equivariant, is just a matter of noticing that, for each $\delta \in \Gamma_{n}$,

$$
\delta \cdot f_{s}\left(F, \phi,\left[\varpi_{n}\right]\right)=\left[F \otimes \pi^{*} L_{\delta}, \phi, \varpi_{n}\right]=f_{s}\left(\delta \cdot\left(F, \phi,\left[\varpi_{n}\right]\right)\right)
$$

because $\mathbb{Z}_{n}$ acts trivially on $\pi^{*} L_{\delta}$.
The action of the Galois group $\mathbb{Z}_{n}$ on the product $\mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times S_{n}^{|D|}$ can therefore be absorbed in the $S_{n}^{|D|}$ factor.

### 6.2. Calculation of the polynomial.

Proposition 6.6. For any non-trivial $\gamma \in \Gamma_{n}$, we have the following isomorphism regarding the $\Gamma_{n}$-invariant part of $H_{c}^{*}\left(\mathcal{M}^{\gamma}, \mathbb{C}\right)$ :

$$
H_{c}^{*}\left(\mathcal{M}^{\gamma}, \mathbb{C}\right)^{\Gamma_{n}} \cong H^{*}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right) \otimes H_{c}^{*}\left(\mathbb{C}^{(n-1)(g-1)}, \mathbb{C}\right) \otimes H^{*}\left(S_{n}^{|D|} / \mathbb{Z}_{n}, \mathbb{C}\right)
$$

Proof. By Proposition 6.5, we can consider the $\Gamma_{n}$-action on $\mathrm{Nm}_{\pi}^{-1}(\Lambda) \times \mathbb{C}^{(n-1)(g-1)} \times$ $\left(S_{n}^{|D|} / \mathbb{Z}_{n}\right)$, where $\Gamma_{n}$ acts trivially on the second and third factors, hence the corresponding cohomologies are $\Gamma_{n}$-invariant. It then suffices to prove that $H^{*}\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda), \mathbb{C}\right)^{\Gamma_{n}} \cong$ $H^{*}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)$.

Consider the symplectic basis of $\Gamma_{n}$ given by (5.15). By Proposition 5.13, the subgroup generated by $\delta_{0}$ acts freely and transitively on $\pi_{0}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda)\right)$, while any $\delta \notin\left\langle\delta_{0}\right\rangle$ acts trivially on these components. Write the decomposition of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$ into connected components as

$$
\begin{equation*}
\operatorname{Nm}_{\pi}^{-1}(\Lambda)=N_{1} \sqcup \cdots \sqcup N_{n} \tag{6.14}
\end{equation*}
$$

where the indices are chosen so that if $x \in N_{i}$ then $\delta_{0}(x) \in N_{i+1}$ (where we assume $n+1=1$ ). Each $N_{i}$ is a torsor for the Prym variety of $X_{\gamma}$, hence their cohomologies are the same.

Take a cohomology class in $H^{k}\left(\operatorname{Nm}_{\pi}^{-1}(\Lambda), \mathbb{C}\right)$ represented by a $k$-form $\omega \in \mathcal{A}^{*}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda), \mathbb{C}\right)$. Write

$$
\begin{equation*}
\omega=\left(\omega_{1}, \ldots, \omega_{i}, \ldots, \omega_{n}\right) \tag{6.15}
\end{equation*}
$$

according to (6.14), where each $\omega_{i}$ represents a cohomology class in $H^{k}\left(N_{i}, \mathbb{C}\right)$. The action of $\delta \in \Gamma_{n}$ on the cohomology class represented by $\omega$ is given by pullback

$$
\begin{equation*}
\delta \cdot \omega=\delta^{*} \omega \tag{6.16}
\end{equation*}
$$

So the decomposition of $\delta_{0} \cdot \omega$ in (6.14) is given by

$$
\begin{equation*}
\delta_{0} \cdot \omega=\left(\delta_{0}^{*} \omega_{2}, \ldots, \delta_{0}^{*} \omega_{i+1}, \ldots, \delta_{0}^{*} \omega_{1}\right) \tag{6.17}
\end{equation*}
$$

By (6.15) and (6.17) we see that the class represented by $\omega$ is invariant by the subgroup $\left\langle\delta_{0}\right\rangle$ if and only if the forms $\omega_{i}$ are such that $\omega_{i}=\delta_{0}^{*} \omega_{i+1}$, that is, if and only if $\omega$ is given by

$$
\begin{equation*}
\omega=\left(\left(\delta_{0}^{n-1}\right)^{*} \omega_{n}, \ldots,\left(\delta_{0}^{n-i}\right)^{*} \omega_{n}, \ldots, \omega_{n}\right) \tag{6.18}
\end{equation*}
$$

Notice that this makes sense because $\delta_{0}$ has order $n$.
Consider now an element $\delta \in \Gamma_{n}$ which is not in the subgroup generated by $\delta_{0}$. Then $\delta$ preserves the connected components $N_{i}$ of $\mathrm{Nm}_{\pi}^{-1}(\Lambda)$, acting hence on each $H^{*}\left(N_{i}, \mathbb{C}\right)$. Each $N_{i}$ is a torsor for the Prym of $X_{\gamma}$ and $\delta$ acts on $N_{i}$ by translations by an element of the Prym:

$$
\delta \cdot M=M \otimes \pi^{*} L_{\delta} .
$$

To see that indeed $\pi^{*} L_{\delta} \in \operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$, note first that it is in the kernel of $\mathrm{Nm}_{\pi}$. Hence it is of the form $\pi^{*} L_{\delta}=F^{-1} \otimes \xi^{*} F$, with $\xi^{\operatorname{deg}(F)}=\langle\delta, \gamma\rangle$, by Proposition 5.12, But $\langle\delta, \gamma\rangle=1$ i.e. $\operatorname{deg}(F)=0$ and $\pi^{*} L_{\delta} \in \operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$ by Proposition 5.6. Since $\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)$ is an abelian variety, every class in $H^{*}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)$ contains a unique representative which is invariant under translations. This property goes through $H^{*}\left(N_{n}, \mathbb{C}\right) \cong H^{*}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)$ considering the torsor structure of $N_{n}$. This means that we can assume that the form $\omega_{n}$ in (6.18) is invariant under translations, so is $\delta$-invariant, i.e., $\delta^{*} \omega_{n}=\omega_{n}$. Hence the action (6.16) of $\delta$ on $\omega$ given by (6.18) is

$$
\delta \cdot \omega=\left(\left(\delta_{0}^{n-1}\right)^{*} \omega_{n}, \ldots,\left(\delta_{0}^{n-i}\right)^{*} \omega_{n}, \ldots, \omega_{n}\right)
$$

We thus conclude that $H^{*}\left(\mathrm{Nm}_{\pi}^{-1}(\Lambda), \mathbb{C}\right)^{\Gamma_{n}}$ is given precisely by the classes represented by the forms of type (6.18). Mapping those to $\left[\omega_{n}\right]$ gives an isomorphism with $H^{*}\left(N_{n}, \mathbb{C}\right)$, hence also with $H^{*}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)$.

Now we can finally compute the sum of the stringy $E$-polynomial of $\mathcal{M} / \Gamma_{n}$ corresponding to non-trivial elements of $\Gamma_{n}$.
Proposition 6.7. For any $n$ prime, the following holds:
$\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}=\frac{n^{2 g}-1}{n}(n!)^{|D|}(u v)^{\left(n^{2}-1\right)(g-1)+|D| n(n-1) / 2}((1-u)(1-v))^{(n-1)(g-1)}$.

Proof. By Proposition 6.6, and since $E\left(\mathbb{C}^{(n-1)(g-1)}\right)=(u v)^{(n-1)(g-1)}$,

$$
E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}=(u v)^{(n-1)(g-1)} E\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right) E\left(S_{n}^{|D|} / \mathbb{Z}_{n}\right),
$$

for each $\gamma \in \Gamma_{n} \backslash\{e\}$.
The polynomial $E\left(S_{n}^{|D|} / \mathbb{Z}_{n}\right)$ is just the constant $\frac{1}{n}(n!)^{|D|}$, i.e., the number of elements of the space $S_{n}^{|D|} / \mathbb{Z}_{n}$.

Being an abelian variety, the cohomology of the Prym of $X_{\gamma}$ is the alternating algebra on $H^{1}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)=H^{0,1}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right) \oplus H^{1,0}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right)$. Write $V=H^{0,1}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right)$ and note that $\operatorname{dim}(V)=\operatorname{dim}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right)=(n-1)(g-1)$. Thus $H^{k}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right), \mathbb{C}\right)=$ $\Lambda^{k}(V \oplus \bar{V})$, therefore

$$
H^{p, q}\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right)=\Lambda^{p} V \otimes \Lambda^{q} \bar{V}
$$

whose dimension is $(\underset{p}{(n-1)(g-1)})\binom{(n-1)(g-1)}{q}$. Hence

$$
\begin{aligned}
E\left(\operatorname{Prym}_{\pi}\left(X_{\gamma}\right)\right) & =\sum_{p, q=0}^{(n-1)(g-1)}(-1)^{p+q}\binom{(n-1)(g-1)}{p}\binom{(n-1)(g-1)}{q} u^{p} v^{q} \\
& =((1-u)(1-v))^{(n-1)(g-1)}
\end{aligned}
$$

We are now left to the computation of the fermionic shift $F(\gamma)$ as defined in (3.12). From (3.13), we know that $F(\gamma)=\operatorname{dim}\left(N_{p} \mathcal{M}^{\gamma}\right) / 2$, but from (2.8) and (6.12), we conclude that

$$
\begin{equation*}
F(\gamma)=n(n-1)(g-1+|D| / 2) \tag{6.19}
\end{equation*}
$$

Therefore, for each $\gamma \neq e$,

$$
E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}=\frac{1}{n}(n!)^{|D|}(u v)^{(n-1)(g-1)+n(n-1)(g-1+|D| / 2)}((1-u)(1-v))^{(n-1)(g-1)} .
$$

This is independent of $\gamma \in \Gamma_{n} \backslash\{e\}$, thus summing up this expression for all non-trivial elements of $\Gamma_{n}$, yields
$\sum_{\gamma \neq e} E\left(\mathcal{M}^{\gamma}\right)^{\Gamma_{n}}(u v)^{F(\gamma)}=\frac{n^{2 g}-1}{n}(n!)^{|D|}(u v)^{\left(n^{2}-1\right)(g-1)+|D| n(n-1) / 2}((1-u)(1-v))^{(n-1)(g-1)}$,
as claimed.

## References

[1] D. Alfaya, T. Gómez, Torelli theorem for the parabolic Deligne-Hitchin moduli space, J. Geom. Phys. 123 (2018), 448-462.
[2] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, Geometry of Algebraic Curves. Vol. I., Grundlehren Math. Wiss. 267, Springer-Verlag, 1985.
[3] M. F. Atiyah, R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1982), 523-615.
[4] V. Batyrev, Stringy Hodge numbers of varieties with Gorenstein canonical singularities, in Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), 1-32, World Sci. Publ., River Edge, NJ, 1998.
[5] V. Batyrev, D. Dais, Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology 35 (1996), 901-929.
[6] O. Biquard, O. García-Prada, I. Mundet i Riera, Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group, arXiv:1510.04207 [math.AG] (2015).
[7] I. Biswas, A. Dey, SYZ duality for parabolic Higgs moduli spaces, Nuclear Physics B 862 (2012), 327-340.
[8] I. Biswas, S. Ramanan, An infinitesimal study of the moduli of Hitchin pairs, J. London Math. Soc. (2) 49 (1994), 219-231.
[9] H. Boden, K. Yokogawa, Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves. I, Internat. J. Math. 7 (1996), 573-598.
[10] F. Bottacin, Symplectic geometry on moduli spaces of stable pairs, Ann. Sci. Ec. Norm. Super., IV. Ser. 28 (1995), 391-433.
[11] W. Chen, Y. Ruan, A new cohomology theory of orbifold, Comm. Math. Phys. 248 (2004), 1-31.
[12] R. Donagi, T. Pantev, Langlands duality for Hitchin systems, Invent. Math. 189 (2012), no. 3, 653-735.
[13] O. García-Prada, P. Gothen, V. Muñoz, Betti numbers of the moduli space of rank 3 parabolic Higgs bundles, Mem. Amer. Math. Soc. 187 (2007), no. 879, viii+80 pp.
[14] T. L. Gómez, M. Logares, A Torelli theorem for the moduli space of parabolic Higgs bundles, Adv. Geom. 11 (2011), 429-444.
[15] P. B. Gothen, The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface, Int. J. Math. 5 (1994), 861-875.
[16] M. Groechenig, D. Wyss, and P. Ziegler, Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration, arXiv:1707.06417 [math.AG] (2017).
[17] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, 52, Springer-Verlag, 1977.
[18] T. Hausel, F. Rodriguez-Villegas, Cohomology of large semiprojective hyperkähler varieties, Astérisque 370 (2015), 113-156.
[19] T. Hausel, M. Thaddeus, Examples of mirror partners arising from integrable systems, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 313-318.
[20] T. Hausel, M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, Invent. Math. 153 (2003), 197-229.
[21] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1987), 59-126.
[22] N. J. Hitchin, Stable bundles and integrable systems, Duke Math. J. 54 (1987), 91-114.
[23] N. J. Hitchin, Lie groups and Teichmüller space, Topology 31 (1992), 449-473.
[24] N. J. Hitchin, Lectures on special Lagrangian submanifolds, in: Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, Cambridge, MA, 1999, in: AMS/IP Stud. Adv. Math. 23, Amer. Math. Soc., Providence (2001), 151-182.
[25] Y. I. Holla, Poincaré polynomial of the moduli space of parabolic bundles, Proc. Indian Acad. Sci 110 (2000), 233-261.
[26] H. Konno, Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface, J. Math. Soc. Japan 45 (1993), 253-276.
[27] S. Lang, On quasi-algebraic closure, Ann. of Math. (2) 55 (1952), 373-390.
[28] M. Logares, J. Martens, Moduli of parabolic Higgs bundles and Atiyah algebroids, J. Reine Angew. Math. 649 (2010), 89-116.
[29] E. Markman, Spectral curves and integrable systems, Compositio Math. 93 (1994), 255-290.
[30] D. Mumford, Theta characteristics of an algebraic curve, Ann. Ecole Norm. Sup. (4) 4 (1971), 181-192.
[31] M. S. Narasimhan, S. Ramanan, Generalised Prym varieties as fixed points, J. Indian Math. Soc. 39 (1975), 1-19.
[32] B. Nasatyr, B. Steer, Orbifold Riemann surfaces and the Yang-Mills-Higgs equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 22 (1995), 595-643.
[33] N. Nitsure, Cohomology of the moduli space of parabolic vector bundles, Proc. Indian Acad. Sci 95 (1986), 61-77.
[34] A. Polishchuk, Abelian varieties, theta functions and the Fourier transform, Cambridge Tracts in Mathematics 153, Cambridge University Press, Cambridge, 2003.
[35] C.S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque 96, Société Mathématique de France, Paris, 1982.
[36] C. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (1990), 713-770.
[37] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety II, Inst. Hautes Études Sci. Publ. Math. 80 (1994), 5-79.
[38] A. Strominger, S.-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B 479 (1996), 243-259.
[39] M. Thaddeus, Variation of moduli of parabolic Higgs bundles, J. Reine Angew. Math. 547 (2002), 1-14.
[40] K. Yokogawa, Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves, J. Math. Kyoto Univ. 33 (1993), 451-504.
[41] K. Yokogawa, Infinitesimal deformation of parabolic Higgs sheaves, Internat. J. Math. 6 (1995), 125-148.

## Peter B. Gothen

Centro de Matemática da Universidade do Porto, CMUP
Faculdade de Ciências, Universidade do Porto
Rua do Campo Alegre 687, 4169-007 Porto, Portugal
www.fc.up.pt
email: pbgothen@fc.up.pt

## André G. Oliveira

Centro de Matemática da Universidade do Porto, CMUP
Faculdade de Ciências, Universidade do Porto
Rua do Campo Alegre 687, 4169-007 Porto, Portugal
www.fc.up.pt
email: andre.oliveira@fc.up.pt
On leave from:
Departamento de Matemática, Universidade de Trás-os-Montes e Alto Douro, UTAD
Quinta dos Prados, 5000-911 Vila Real, Portugal
www.utad.pt
email: agoliv@utad.pt


[^0]:    Date: 29 August 2018.
    2010 Mathematics Subject Classification. 14H60 (Primary); 14H40, 14H70 (Secondary).
    This work was partially supported by CMUP (UID/MAT/00144/2013) and the project PTDC/MATGEO/2823/2014 funded by FCT (Portugal) with national funds. Second author was also partially supported by the Post-Doctoral fellowship SFRH/BPD/100996/2014 funded by FCT (Portugal) with national funds. The authors acknowledge support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).

[^1]:    ${ }^{1}$ In Section 5 we shall need to consider the norm map and corresponding Prym of unramified covers.
    ${ }^{2}$ The authors thank Laura Fredrickson for pointing out a mistake at this point of an earlier version of the paper.

