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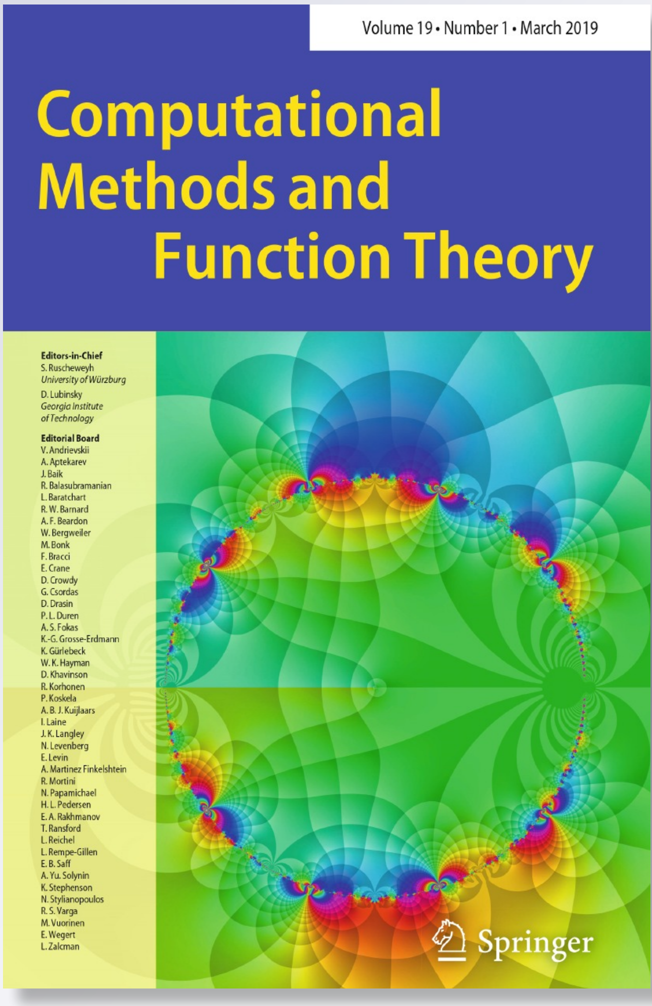

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Bicomplex k-Pell Quaternions

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Abstract
The aim of this work is to consider the bicomplex k-Pell quaternions and to present some properties involving this sequence, including the Binet-style formulae and the generating functions. Furthermore, Cassini's identity, Catalan's identity, and d'Ocagne's identity for this type of bicomplex quaternions are given, and a different way to find the nth term of this sequence is stated using the determinant of a tridiagonal matrix whose entries are bicomplex k-Pell quaternions.

Keywords Bicomplex number · k-Pell number · k-Pell quaternion · Bicomplex quaternion · Bicomplex k-Pell quaternion

Mathematics Subject Classification 11B39 · 20G20 · 11R52

1 Introduction and Background

The set of bicomplex numbers, denoted by $\mathbb{BC}$, forms a two-dimensional algebra over $\mathbb{C}$, and since $\mathbb{C}$ is of dimension two over $\mathbb{R}$, the bicomplex numbers are an algebra over $\mathbb{R}$ of dimension four. The bicomplex numbers are defined by the basis $1, i, j, ij$, where i, j, and ij satisfy the properties:

$$i^2 = -1,\ j^2 = -1,\ ij = ji. \tag{1}$$

A bicomplex number x can be expressed as follows:

$$x = x_1 + ix_2 + jx_3 + ijx_4 = (x_1 + ix_2) + j(x_3 + ix_4),$$

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where $x_1,\ x_2,\ x_3,\ x_4 \in \mathbb{R}$.

The bicomplex numbers share some structures and properties of the complex numbers $\mathbb{C}$, but there are differences between them [17]. Bicomplex numbers forms a commutative ring with unity which contain the complex numbers.

For any $x = x_1 + ix_2 + jx_3 + ijx_4$ and $y = y_1 + iy_2 + jy_3 + ijy_4$, bicomplex addition and product are defined by the following:

$$x + y = (x_1 + y_1) + i\,(x_2 + y_2) + j\,(x_3 + y_3) + ij\,(x_4 + y_4)$$

and

$$\begin{aligned} x \times y = (x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4) + i\,(x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3) \\ + j\,(x_1y_3 + x_3y_1 - x_2y_4 - x_4y_2) + ij\,(x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2)\,, \end{aligned}$$

respectively. The set of bicomplex numbers $\mathbb{BC}$ is a real vector space with this addition and the multiplication of a bicomplex number by a real scalar, and with the bicomplex number product, $\times$ is a real associative algebra. In addition, the vector space with the properties of scalar multiplication and the product of the bicomplex numbers is a commutative algebra. For more details about these type of numbers, see, for example, the works [18,20], among others.

There are three different conjugations for bicomplex numbers as follows:

$$\begin{aligned} x_i^* &= x_1 - ix_2 + jx_3 - ijx_4 = (x_1 - ix_2) + j\,(x_3 - ix_4)\,, \\ x_j^* &= x_1 + ix_2 - jx_3 - ijx_4 = (x_1 + ix_2) - j\,(x_3 + ix_4)\,, \\ x_{ij}^* &= x_1 - ix_2 - jx_3 + ijx_4 = (x_1 - ix_2) - j\,(x_3 - ix_4)\,, \end{aligned}$$

and the squares of norms of the bicomplex numbers are given by

$$\begin{aligned} N_{x_i}^2 &= |x_1^2 + x_2^2 - x_3^2 - x_4^2 + 2j\,(x_1x_3 + x_2x_4)\,|, \\ N_{x_j}^2 &= |x_1^2 - x_2^2 + x_3^2 - x_4^2 + 2i\,(x_1x_2 + x_3x_4)\,|, \\ N_{x_{ij}}^2 &= |x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2ij\,(x_1x_4 - x_2x_3)\,|. \end{aligned}$$

Quaternions were formally introduced by W. R. Hamilton in 1843 and some background about this type of hypercomplex numbers can be found, for example, in [8,26]. The field $\mathbb{H}$ of quaternions is a four-dimensional non-commutative $\mathbb{R}-$field generated by four base elements $1,\ i,\ j$ and l that satisfy the following rules:

$$i^2 = j^2 = l^2 = ijl = -1,\ ij = -ji = l,\ jl = -lj = i,\ li = -il = j. \quad (2)$$

Quaternions and bicomplex numbers are generalizations of complex numbers, but one difference between them is that quaternions are non-commutative, whereas bicomplex numbers are commutative. Similarly, considering bicomplex quaternions, they can also be defined by four base elements $1,\ i,\ j$, and ij that satisfy the rules (1).

In [12], Horadam introduced the Fibonacci quaternion sequence and such sequences have been studied in several papers (see, for example, [11,13–15]). Also generalizations of the Fibonacci quaternions have been presented in the literature (see, for example, [1,9,19,22]).

For any positive integer number k, generalizations of certain sequences of positive integers have been studied by many researchers. For example, the study of the k-Pell sequence appears in [5,6], among others. Recall that, for any positive integer number k, the sequence of k-Pell numbers $\{P_{k,n}\}_{n=0}^{\infty}$ is defined by the following:

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ n \geq 2 \tag{3}$$

with the initial terms $P_{k,0} = 0$ and $P_{k,1} = 1$.

The Binet-style formula for this sequences is given by the following:

$$P_{k,n} = \frac{(s_1)^n - (s_2)^n}{s_1 - s_2}, \tag{4}$$

where $s_1 = 1 + \sqrt{1+k}$ and $s_2 = 1 - \sqrt{1+k}$ are the roots of the characteristic equation $s^2 - 2s - k = 0$ associated with the above recurrence relation (3). Note that $s_1 + s_2 = 2,\ s_1 s_2 = -k$ and $s_1 - s_2 = 2\sqrt{1+k}$.

The quaternion version of Pell sequence and some generalizations have been considered by several researchers in their works (see, for example, [4,7,21,23]).

The more recent research in the topic of sequences of bicomplex quaternions is the work of Aydın in [2] about the bicomplex Fibonacci quaternions. Motivated essentially by that work, in this paper, we introduce the bicomplex k-Pell quaternions and we obtain some properties, including the respective Binet-style formulae, generating functions and some other identities.

2 The Bicomplex k-Pell Quaternions and Some Basic Properties

Let $1,\ i,\ j$, and ij satisfy the rules (1). Let k be a positive integer.

Definition 1 The bicomplex k-Pell quaternions $\{\mathrm{BC}_{k,n}^{P}\}_{n=0}^{\infty}$ are defined by the following:

$$\mathrm{BC}_{k,n}^{P} = P_{k,n} + i\,P_{k,n+1} + j\,P_{k,n+2} + ij\,P_{k,n+3},$$

where $P_{k,n}$ is the nth k-Pell number.

From Definition 1 and the use of (3), we easily show that $\{\mathrm{BC}_{k,n}^{P}\}_{n=0}^{\infty}$ can also be defined by the recurrence relation:

$$\mathrm{BC}_{k,n+1}^{P} = 2\mathrm{BC}_{k,n}^{P} + kBC_{k,n-1}^{P},\ n \geq 1 \tag{5}$$

with the initial conditions $\mathrm{BC}_{k,0}^{P} = i + j\,(2) + ij\,(4+k)$ and $\mathrm{BC}_{k,1}^{P} = 1 + i\,(2) + j\,(4+k) + ij\,(8+4k)$.

For two bicomplex k-Pell quaternions $\mathrm{BC}^{P}_{k,n}$ and $\mathrm{BC}^{P}_{k,m}$, addition and subtraction are obviously defined by the following:

$$\begin{aligned}\mathrm{BC}^{P}_{k,n} \pm \mathrm{BC}^{P}_{k,m} = & \left(P_{k,n} \pm P_{k,m}\right) + i\left(P_{k,n+1} \pm P_{k,m+1}\right)\\ & + j\left(P_{k,n+2} \pm P_{k,m+2}\right) + ij\left(P_{k,n+3} \pm P_{k,m+3}\right),\end{aligned}$$

and multiplication by

$$\begin{aligned}\mathrm{BC}^{P}_{k,n} \times \mathrm{BC}^{P}_{k,m} = & \left(P_{k,n}P_{k,m} - P_{k,n+1}P_{k,m+1} - P_{k,n+2}P_{k,m+2} + P_{k,n+3}P_{k,m+3}\right)\\ & + i\left(P_{k,n}P_{k,m+1} + P_{k,n+1}P_{k,m} - P_{k,n+2}P_{k,m+3} - P_{k,n+3}P_{k,m+2}\right)\\ & + j\left(P_{k,n}P_{k,m+2} + P_{k,n+2}P_{k,m} - P_{k,n+1}P_{k,m+3} - P_{k,n+3}P_{k,m+1}\right)\\ & + ij\left(P_{k,n}P_{k,m+3} + P_{k,n+3}P_{k,m} + P_{k,n+1}P_{k,m+2} + P_{k,n+2}P_{k,m+1}\right)\\ = & \ \mathrm{BC}^{P}_{k,m} \times \mathrm{BC}^{P}_{k,n}.\end{aligned}$$

The multiplication of a bicomplex k-Pell quaternion by the real scalar λ is defined by the following:

$$\lambda \mathrm{BC}^{P}_{k,n} = \lambda P_{k,n} + i\lambda P_{k,n+1} + j\lambda P_{k,n+2} + ij\lambda P_{k,n+3}.$$

The different conjugations for bicomplex k-Pell quaternions are presented as follows:

$$\left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{i} = P_{k,n} - iP_{k,n+1} + jP_{k,n+2} - ijP_{k,n+3} \tag{6}$$

$$\left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{j} = P_{k,n} + iP_{k,n+1} - jP_{k,n+2} - ijP_{k,n+3} \tag{7}$$

$$\left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{ij} = P_{k,n} - iP_{k,n+1} - jP_{k,n+2} + ijP_{k,n+3}. \tag{8}$$

Using these conjugations, we have two basic properties of the bicomplex k-Pell quaternions.

Lemma 1 *For any positive integer number k, the following relations between the conjugate of these bicomplex k-Pell quaternions are true:*

1. $\left(BC^{P}_{k,n} \times \mathrm{BC}^{P}_{k,m}\right)^{*}_{i} = \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{i} \times \left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{i} = \left(BC^{P}_{k,n}\right)^{*}_{i} \times \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{i},$
2. $\left(\mathrm{BC}^{P}_{k,m} \times \mathrm{BC}^{P}_{k,n}\right)^{*}_{j} = \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{j} \times \left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{j} = \left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{j} \times \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{j},$
3. $\left(\mathrm{BC}^{P}_{k,m} \times \mathrm{BC}^{P}_{k,n}\right)^{*}_{ij} = \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{ij} \times \left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{ij} = \left(\mathrm{BC}^{P}_{k,n}\right)^{*}_{ij} \times \left(\mathrm{BC}^{P}_{k,m}\right)^{*}_{ij}.$

Proof We can prove these equalities using (6), (7), and (8), and the multiplication of two bicomplex k-Pell quaternions. □

Lemma 2 *For any positive integer number k, the squares of norms in different ways of the bicomplex k-Pell quaternions are given by the following:*

1. $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_i} = |A_1 + 2jB_1|$,
2. $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_j} = |A_2 + 2iB_2|$,
3. $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_{ij}} = |A_3 + 2ijB_3|$,

where

$$\begin{aligned}
A_1 &= \left(1-5k^2\right) P^2_{k,n} - \left(3+(4+k)^2\right) P^2_{k,n+1} - 4k\,(5+k)\, P_{k,n+1}P_{k,n},\\
B_1 &= kP^2_{k,n} + (4+k)\, P^2_{k,n+1} + 2\,(1+k)\, P_{k,n+1}P_{k,n},\\
A_2 &= \left(1-3k^2\right) P^2_{k,n} + \left(3-(4+k)^2\right) P^2_{k,n+1} - 4k\,(3+k)\, P_{k,n+1}P_{k,n},\\
B_2 &= 2k^2P^2_{k,n} + 2\,(4+k)\, P^2_{k,n+1} + \left(1+8k+k^2\right) P_{k,n+1}P_{k,n},\\
A_3 &= \left(1+5k^2\right) P^2_{k,n} + \left(5+(4+k)^2\right) P^2_{k,n+1} + (4k\,(5+k))\, P_{k,n+1}P_{k,n},\\
B_3 &= 2kP^2_{k,n} - 2P^2_{k,n+1} + 4P_{k,n+1}P_{k,n}.
\end{aligned}$$

Proof These equalities can easily be proved using Definition 1 and the definition of norm of bicomplex k-Pell quaternions taking into account the three different conjugations given in (6), (7), and (8), considering $x_1 = P_{k,n},\ x_2 = P_{k,n+1},\ x_3 = P_{k,n+2}$ and $x_4 = P_{k,n+3}$. We present the proof of the first equality. For this, by taking into account that $P_{k,n+2} = 2P_{k,n+1} + kP_{k,n}$ and $P_{k,n+3} = (4+k)\, P_{k,n+1} + 2kP_{k,n}$, we obtain

$$\begin{aligned}
&P^2_{k,n} + P^2_{k,n+1} - P^2_{k,n+2} - P^2_{k,n+3} + 2j\left(P_{k,n}P_{k,n+2} + P_{k,n+1}P_{k,n+3}\right)\\
&\quad= P^2_{k,n} + P^2_{k,n+1} - \left(2P_{k,n+1} + kP_{k,n}\right)^2 - \left((4+k)\, P_{k,n+1} + 2kP_{k,n}\right)^2 + 2j\,(\cdots)\\
&\quad= \left(1-5k^2\right) P^2_{k,n} - \left(3+(4+k)^2\right) P^2_{k,n+1} - 4k\,(5+k)\, P_{k,n+1}P_{k,n} + 2j\,(\cdots)\\
&\quad= (\cdots) + 2j\left(P_{k,n}\left(2P_{k,n+1} + kP_{k,n}\right) + P_{k,n+1}\left((4+k)\, P_{k,n+1} + 2kP_{k,n}\right)\right)\\
&\quad= (\cdots) + 2j\left(kP^2_{k,n} + (4+k)\, P^2_{k,n+1} + 2\,(1+k)\, P_{k,n+1}P_{k,n}\right),
\end{aligned}$$

and the result follows for $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_i}$.

In a similar way, we can find the expressions of $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_j}$ and $N^2_{\left(\mathrm{BC}^P_{k,n}\right)_{ij}}$. □

The sum of the first n terms of the bicomplex k-Pell quaternions sequence is stated in the next result.

Theorem 1 *The sum of the first n terms of the bicomplex k-Pell quaternions sequence is given by the following:*

$$\sum_{t=0}^{n} \mathrm{BC}^{P}_{k,t} = \left(A + P_{k,n+2}\right) + iA + jA + ij\left(A + P_{k,n+2} + P_{k,n+3}\right),$$

where $A = \frac{1}{k+1}\left(-1 + kP_{k,n} + (k+2)\,P_{k,n+1}\right)$ *denotes the sum of the first n terms of k-Pell sequence, which is given by [24, Prop. 3 item 3].*

Proof By the use of Definition 1, we have:

$$\sum_{t=0}^{n} \mathrm{BC}^{P}_{k,t} = \sum_{t=0}^{n} P_{k,t} + i\sum_{t=0}^{n} P_{k,t+1} + j\sum_{t=0}^{n} P_{k,t+2} + ij\sum_{t=0}^{n} P_{k,t+3}.$$

Now, taking into account the result stated in [24, Prop. 3, item 3] and the initial condition $P_{k,0}$, the result easily follows. □

3 Generating functions and Binet's formula

Next, we shall give the generating functions for the bicomplex k-Pell quaternions sequence. We shall write such sequence as a power series where each term of the sequence corresponds to coefficients of the series. Consider the bicomplex k-Pell quaternions sequence $\{\mathrm{BC}^{P}_{k,n}\}_{n=0}^{\infty}$. By definition of generating function of a sequence, considering this sequence, the associated generating function $g_{\mathrm{BC}^{P}_{k,n}}(t)$ is defined, respectively, by the following:

$$g_{\mathrm{BC}^{P}_{k,n}}(t) = \sum_{n=0}^{\infty} \mathrm{BC}^{P}_{k,n} t^{n}. \tag{9}$$

Therefore, using (9), we obtain the following result:

Theorem 2 *The generating function for the bicomplex k-Pell quaternions sequence is given by the following:*

$$g_{\mathrm{BC}^{P}_{k,n}}(t) = \frac{\mathrm{BC}^{P}_{k,0} + (\mathrm{BC}^{P}_{k,1} - 2BC^{P}_{k,0})t}{1 - 2t - kt^{2}}.$$

Proof 1. Using (9), we have

$$g_{\mathrm{BC}^{P}_{k,n}}(t) = \mathrm{BC}^{P}_{k,0} + \mathrm{BC}^{P}_{k,1}t + \mathrm{BC}^{P}_{k,2}t^{2} + \cdots + \mathrm{BC}^{P}_{k,n}t^{n} + \cdots. \tag{10}$$

Multiplying both sides of (10) by $-2t$, we obtain

$$-2tg_{\mathrm{BC}^{P}_{k,n}}(t) = -2\mathrm{BC}^{P}_{k,0}t - 2\mathrm{BC}^{P}_{k,1}t^{2} - 2\mathrm{BC}^{P}_{k,2}t^{3} - \cdots - 2\mathrm{BC}^{P}_{k,n}t^{n+1} - \cdots. \tag{11}$$

Now, multiplying both sides of (10) by $-kt^2$, we get

$$-kg_{\mathrm{BC}^P_{k,n}}(t)t^2 = -k\mathrm{BC}^P_{k,0}t^2 - k\mathrm{BC}^P_{k,1}t^3 - k\mathrm{BC}^P_{k,2}t^4 - \cdots - k\mathrm{BC}^P_{k,n}t^{n+2} - \cdots. \quad (12)$$

Adding (10), (11), (12), and using (5), we have

$$(1-2t-kt^2)g_{\mathrm{BC}^P_{k,n}}(t) = \mathrm{BC}^P_{k,0} + (\mathrm{BC}^P_{k,1} - 2BC^P_{k,0})t,$$

and the result follows. □

The following result, with easy proof, uses Binet's formula of $P_{k,n}$ given by (4) and it will be useful in the statement of the Binet formula of $BC^P_{k,n}$.

Lemma 3 *For the k-Pell number, we have*

1. $P_{k,l+1} - s_2 P_{k,l} = s_1^l$
2. $P_{k,l+1} - s_1 P_{k,l} = s_2^l$,

where s_1 and s_2 are the roots of the characteristic equation associated with the above recurrence relation (3).

The next result will be also used in the statement of the Binet formula for the bicomplex k-Pell quaternion sequence.

Theorem 3 *For the generating function given in Theorem 2, we have*

$$g_{\mathrm{BC}^P_{k,n}}(t) = \frac{1}{s_1 - s_2}\left(\frac{\mathrm{BC}^P_{k,1} - s_2\mathrm{BC}^P_{k,0}}{1 - s_1 t} - \frac{\mathrm{BC}^P_{k,1} - s_1\mathrm{BC}^P_{k,0}}{1 - s_2 t}\right).$$

Proof From the expression of $g_{\mathrm{BC}^P_{k,n}}(t)$ given by Theorem 2 and by the use of the fact that $s_1 + s_2 = 2,\ s_1 s_2 = -k$, we have

$$\begin{aligned} g_{\mathrm{BC}^P_{k,n}}(t) &= \frac{\mathrm{BC}^P_{k,0} + (\mathrm{BC}^P_{k,1} - 2\mathrm{BC}^P_{k,0})t}{1 - 2t - kt^2} \\ &= \frac{\mathrm{BC}^P_{k,0} + (\mathrm{BC}^P_{k,1} - 2\mathrm{BC}^P_{k,0})t}{1 - (s_1 + s_2)\,t + s_1 s_2 t^2} \\ &= \frac{\mathrm{BC}^P_{k,0} + (\mathrm{BC}^P_{k,1} - 2\mathrm{BC}^P_{k,0})t}{(1 - s_1 t)\,(1 - s_2 t)}. \end{aligned}$$

Now, multiplying and dividing the right side of this expression by $s_1 - s_2$, the result follows. □

The next result gives the Binet formula for the bicomplex k-Pell quaternion sequence.

Theorem 4 *For $n \geq 0$, we have*

$$\mathrm{BC}^P_{k,n} = \frac{\widehat{s_1}(s_1)^n - \widehat{s_2}(s_2)^n}{s_1 - s_2},$$

where $\widehat{s_1}$ and $\widehat{s_2}$ are bicomplex quaternions defined by $\widehat{s_1} = 1 + is_1 + js_1^2 + ijs_1^3$ and $\widehat{s_2} = 1 + is_2 + js_2^2 + ijs_2^3$, respectively.

Proof 1. Using (3), the Binet formula for the k-Pell numbers considered in (4) and the Definition 1, we have

$$\begin{aligned} g_{\mathrm{BC}^P_{k,n}}(t) &= \frac{1}{s_1 - s_2}\left(\left(\mathrm{BC}^P_{k,1} - s_2\mathrm{BC}^P_{k,0}\right)\sum_{n=0}^{\infty} s_1^n t^n - \left(\mathrm{BC}^P_{k,1} - s_1\mathrm{BC}^P_{k,0}\right)\sum_{n=0}^{\infty} s_2^n t^n\right) \\ &= \frac{1}{s_1 - s_2}\left(\widehat{s_1}\sum_{n=0}^{\infty} s_1^n t^n - \widehat{s_2}\sum_{n=0}^{\infty} s_2^n t^n\right) \\ &= \sum_{n=0}^{\infty}\left(\frac{\widehat{s_1}s_1^n - \widehat{s_2}s_2^n}{s_1 - s_2}\right)t^n, \end{aligned}$$

and by (9), the result easily follows. □

4 Some Identities Involving These Sequences

As a consequence of the Binet formulae of Theorem 4, we get, for the sequence of bicomplex k-Pell quaternions, the following interesting identities.

Proposition 1 (Catalan's identity) *For natural numbers n, r, with $n \geq r$ and a positive integer number k, if $\mathrm{BC}^P_{k,n}$ is the nth bicomplex k-Pell quaternion, then the following identity is true:*

$$\mathrm{BC}^P_{k,n-r}BC^P_{k,n+r} - (\mathrm{BC}^P_{k,n})^2 = (-1)^{n-r+1}\,k^{n-r}\left(P_{k,r}\right)^2\widehat{s_1}\widehat{s_2},$$

where $\widehat{s_1}$ and $\widehat{s_2}$ are the bicomplex k-Pell quaternions defined in Theorem 4.

Proof By the use of the Binet Formula of Theorem 4, the fact that the product of any two bicomplex k-Pell quaternions is commutative and the identities $s_1s_2 = -k$, $\frac{s_1}{s_2} = \frac{(s_1)^2}{-k}$, $\frac{s_2}{s_1} = \frac{(s_2)^2}{-k}$, we have

$$\begin{aligned} \mathrm{BC}^P_{k,n-r}BC^P_{k,n+r} - (\mathrm{BC}^P_{k,n})^2 &= \left(\frac{\widehat{s_1}(s_1)^{n-r} - \widehat{s_2}(s_2)^{n-r}}{s_1 - s_2}\right)\left(\frac{\widehat{s_1}(s_1)^{n+r} - \widehat{s_2}(s_2)^{n+r}}{s_1 - s_2}\right) \\ &\quad - \left(\frac{\widehat{s_1}(s_1)^n - \widehat{s_2}(s_2)^n}{s_1 - s_2}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-k)^n \left(-\widehat{s_1}\widehat{s_2}\left(\frac{s_2}{s_1}\right)^r - \widehat{s_2}\widehat{s_1}\left(\frac{s_1}{s_2}\right)^r + 2\widehat{s_1}\widehat{s_2}\right)}{(s_1 - s_2)^2} \\
&= \frac{(-k)^{n-r}\left(-\widehat{s_1}\widehat{s_2}\,(s_2)^{2r} - \widehat{s_2}\widehat{s_1}\,(s_1)^{2r} + 2\widehat{s_1}\widehat{s_2}(s_1 s_2)^r\right)}{(s_1 - s_2)^2} \\
&= \frac{-\widehat{s_1}\widehat{s_2}(-k)^{n-r}\left((s_1)^{2r} - 2\,(-k)^r + (s_2)^{2r}\right)}{(s_1 - s_2)^2} \\
&= \frac{\widehat{s_1}\widehat{s_2}\,(-1)^{n-r+1}\,k^{n-r}\left((s_1)^r - (s_2)^r\right)^2}{(s_1 - s_2)^2}.
\end{aligned}
$$

Now, using the Binet formula (4), the result follows. □

For $r = 1$ in Catalan's identity, the Cassini identity for this sequence is stated in the next result.

Proposition 2 (Cassini's identity) *For a natural number n and a positive integer k, if $\mathrm{BC}^P_{k,n}$ is the nth bicomplex k-Pell quaternion, then the following identity is true:*

$$
BC^P_{k,n-1} BC^P_{k,n+1} - (BC^P_{k,n})^2 = (-k)^n k^{n-1} \widehat{s_1}\widehat{s_2},
$$

where $\widehat{s_1}$ and $\widehat{s_2}$ are the bicomplex k-Pell quaternions defined in Theorem 4.

The d'Ocagne identity for this sequence can also be obtained as a consequence of the use of the respective Binet formula. We get the following result.

Proposition 3 (d'Ocagne's identity) *Suppose that n is a non-negative integer number and m any natural number. If $m > n$ and $BC^P_{k,n}$ is the nth bicomplex k-Pell quaternion, then the expression of the d'Ocagne's identity is given by the following:*

$$
\mathrm{BC}^P_{k,m} BC^P_{k,n+1} - \mathrm{BC}^P_{k,m+1} BC^P_{k,n} = \widehat{s_1}\widehat{s_2}\,(-k)^n\, P_{k,m-n},
$$

where $\widehat{s_1}$ and $\widehat{s_2}$ are the bicomplex k-Pell quaternions defined in Theorem 4.

Proof Once more, using the Binet Formula of Theorem 4 and the fact that $s_1 s_2 = -k$, we have

$$
\begin{aligned}
&\mathrm{BC}^P_{k,m} BC^P_{k,n+1} - BC^P_{k,m+1} BC^P_{k,n} \\
&= \left(\frac{\widehat{s_1}(s_1)^m - \widehat{s_2}(s_2)^m}{s_1 - s_2}\right)\left(\frac{\widehat{s_1}(s_1)^{n+1} - \widehat{s_2}(s_2)^{n+1}}{s_1 - s_2}\right) \\
&\quad - \left(\frac{\widehat{s_1}(s_1)^{m+1} - \widehat{s_2}(s_2)^{m+1}}{s_1 - s_2}\right)\left(\frac{\widehat{s_1}(s_1)^n - \widehat{s_2}(s_2)^n}{s_1 - s_2}\right) \\
&= \frac{(-k)^n \left(\widehat{s_1}\widehat{s_2}\,(s_1)^{m-n}\,(s_1 - s_2) - \widehat{s_2}\widehat{s_1}\,(s_2)^{m-n}\,(s_1 - s_2)\right)}{(s_1 - s_2)^2} \\
&= \frac{(-k)^n\,(s_1 - s_2)\left(\widehat{s_1}\widehat{s_2}\,(s_1)^{m-n} - \widehat{s_2}\widehat{s_1}\,(s_2)^{m-n}\right)}{(s_1 - s_2)^2}
\end{aligned}
$$

$$= \frac{(-k)^n\, (\widehat{s_1}\widehat{s_2})\, \left((s_1)^{m-n} - (s_2)^{m-n}\right)}{(s_1 - s_2)},$$

and now, using the Binet formula for the k-Pell numbers, the result follows. □

5 Tridiagonal matrix with bicomplex *k*-Pell quaternions

In this section, we present another way to obtain the nth term of the bicomplex k-Pell quaternion sequence as the computation of a tridiagonal matrix. We shall adopt the ideas stated in [16] using the following result which is stated in such work:

Theorem 5 *Let* $\{x_n\}_n$ *be any second-order linear sequence defined recursively by the following:*

$$x_{n+1} = Ax_n + Bx_{n-1},\ n \geq 1,$$

with $x_0 = C,\ x_1 = D$. *Then, for all* $n \geq 0$*:*

$$x_n = \begin{vmatrix} C & D & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & B & 0 & \cdots & 0 & 0 \\ 0 & -1 & A & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A & B \\ 0 & 0 & 0 & 0 & \cdots & -1 & A \end{vmatrix}_{(n+1)\times(n+1)}.$$

In the case of the bicomplex k-Pell quaternions sequence and taking into account the recurrence relation (5), we have $A = 2$, $B = k$, $C = \mathrm{BC}^P_{k,0} = i + j\,(2) + ij\,(4+k)$ and $D = \mathrm{BC}^P_{k,1} = 1 + i\,(2) + j\,(4+k) + ij\,(8+4k)$. Then, by the use of the previous theorem, we have the following result which gives a different way to calculate the nth term of this sequence:

Proposition 4 *For* $n \geq 0$, *we have*

$$\mathrm{BC}^P_{k,n} = \begin{vmatrix} \mathrm{BC}^P_{k,0} & BC^P_{k,1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & k & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & k \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{(n+1)\times(n+1)}.$$

Different types of matrices have been used in the study of several types of sequences (see, for example, [3,10,25], among others). Other kind of tridiagonal matrices can be used to find different ways to calculate the nth term of this sequence as it is stated in [3], with Proposition 1 used for some kind of Fibonacci polynomial sequence.

6 Conclusions

In this paper, the sequence of bicomplex k-Pell quaternions defined by a recurrence relation of second order was introduced. Some properties involving this sequence, including the Binet formula and the generating function, were presented. Note that the results stated in this paper for the particular case of $k = 1$ give us the corresponding results for the sequence of the bicomplex Pell quaternions. Considering tridiagonal matrices whose entries are bicomplex k-Pell quaternions are presented a different way to obtain the nth term of the bicomplex k-Pell quaternions sequence.

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